

# ON THE LOCAL RESIDUE SYMBOL IN THE STYLE OF TATE AND BEILINSON

OLIVER BRAUNLING

ABSTRACT. Tate gave a famous construction of the residue symbol on curves by using some non-commutative operator algebra in the context of algebraic geometry. We explain Beilinson's multidimensional generalization, which is not so well-documented in the literature. As a novelty, we introduce a Hochschild/cyclic homology counterpart of the theory. It allows to bypass relative homology by exploiting an excision phenomenon and this leads to a new construction of the same map. All constructions work readily for non-commutative rings, but I cannot offer much of an interpretation what these generalized maps mean.

Joint work with M. Groechenig and J. Wolfson will allow to phrase this theory in the framework of Tate categories [4], which leads to a more general categorical treatment. For the present paper, we work in the language of the original works.

Suppose  $X/k$  is a smooth proper algebraic curve over a field. One can define the residue of a rational 1-form  $\omega$  at a closed point  $x$  as

$$(0.1) \quad \text{res}_x \omega = \text{Tr}_{\kappa(x)/k} a_{-1}, \quad \text{where} \quad \omega = \sum a_i t^i dt$$

in terms of a local coordinate  $t$ , i.e. by picking an isomorphism  $\text{Frac } \widehat{\mathcal{O}}_{X,x} \simeq \kappa(x)((t))$ . This works, but is unwieldy since it depends on the choice of the isomorphism and one needs to prove that it is well-defined, cf. Serre [31, Ch. II]. One could ask for a bit more:

**Aim:** Construct the local residue symbol without ever needing to choose coordinates.

J. Tate [32] has pioneered an approach which circumvents choices of coordinates at all times by employing ideas in the style of functional analysis: The local field

$$(0.2) \quad \widehat{\mathcal{K}}_{X,x} := \text{Frac } \widehat{\mathcal{O}}_{X,x} = \varinjlim_{s \neq 0} \varprojlim_i \mathcal{O}_{X,x} / \mathfrak{m}_x^i \langle s^{-1} \rangle$$

carries a canonical topology, defined by viewing it as an ind-pro limit of finite-dimensional discrete  $k$ -vector spaces. This topology needs no assumptions on the base field, e.g. it could be just a finite field. We get a non-commutative algebra of continuous  $k$ -vector space endomorphisms  $E$ . Via multiplication operators  $x \mapsto f \cdot x$  the functions  $f \in \widehat{\mathcal{K}}_{X,x}$  embed into  $E$ . Using the ideal of compact operators, Tate shows that  $E$  has a canonical central extension  $\widehat{E}$  as a Lie algebra by a formal element  $c$  so that

$$(0.3) \quad [f, g]_{\widehat{E}} = \text{res}_x f dg \cdot c,$$

Tate now uses the left-hand side as an intrinsically coordinate-independent definition for the residue (R. Hartshorne advertises this as ‘clever’ in [12, Ch. III, §7]). For an  $n$ -dimensional smooth proper algebraic variety  $X/k$  the global residue

$$H^n(X, \Omega_{X/k}^n) \longrightarrow k$$

---

*Date:* April 1, 2014.

This work has been partially supported by the DFG SFB/TR45 “Periods, moduli spaces, and arithmetic of algebraic varieties” and the Alexander von Humboldt Stiftung.

is induced from  $n$ -dimensional local residue symbols. There is the conventional approach to this using A. Grothendieck's residue symbol [11], however A. Beilinson [1] has shown that one can also describe this map by a beautiful multidimensional generalization of Tate's approach. He interprets the commutators which appear in Tate's theory as low-degree avatars of the differential in Lie homology. As such, one can give an explicit formula for the higher residue in terms of cascading commutators, roughly generalizing eq. 0.3.

**0.1. Review of the perspective of Grothendieck-Hartshorne.** Let us first describe the residue symbol from the viewpoint of Grothendieck and Hartshorne [11]: In broadest terms, if  $f : X \rightarrow Y$  is a (very general) morphism between (very general) varieties, there should be an adjunction

$$\mathbf{R}f_! : \mathbf{D}^+ \mathcal{Q}Coh_X \rightleftarrows \mathbf{D}^+ \mathcal{Q}Coh_Y : \mathbf{R}f^!$$

between the respective derived categories of quasi-coherent sheaves, having certain properties. In particular, we get a co-unit natural transformation  $\mathbf{R}f_! \mathbf{R}f^! \rightarrow \text{id}$ . There are many approaches to this subject, with large variations in generality, technical level, explicitness. We will have nothing to say about such a general setup, see [6], [11], [18], [19], [21] for example. We restrict to the classical case, due to Serre [30], of  $f$  being proper smooth of relative dimension  $n$  to a point  $\text{Spec } k$ . Then  $\mathbf{R}f_! = \mathbf{R}f_*$  and  $\mathbf{R}f^! \mathcal{F} = f^* \mathcal{F} \otimes \Omega_{X/k}^n$ . Take  $\mathcal{F} := \mathcal{O}_{\text{Spec } k}$ , then the co-unit boils down to a canonical morphism

$$(0.4) \quad H^n(X, \Omega_{X/k}^n) \longrightarrow k.$$

In order to work with this map explicitly, one needs to pick an acyclic resolution for the sheaf  $\Omega_{X/k}^n$ . Usually one takes the Cousin complex, a flasque resolution,

$$(0.5) \quad \Omega_{X/k}^n \simeq \left[ \coprod_{x \in X^0} H_x^0(X, \Omega_{X/k}^n) \rightarrow \cdots \rightarrow \coprod_{x \in X^n} H_x^n(X, \Omega_{X/k}^n) \right]_{0,n}$$

(see [11, Ch. IV] for details), where  $X^i$  denotes the set of points of  $X$  so that  $\overline{\{x\}}$  has codimension  $i$ , and  $H_x^i$  denotes the  $i$ -th local cohomology group with support  $\{x\}$ . Describing the morphism in eq. 0.4 thus reduces to describing a suitable morphism  $H_x^n(X, \Omega_{X/k}^n) \rightarrow k$ , which is an entirely local consideration (this aspect is explained in various places, e.g. Hartshorne [11], or by Sastry-Yekutieli [29]). Here is a sketch: Local cohomology can be computed on  $X := \text{Spec } \mathcal{O}_{X,x}$  instead of the original scheme  $X$  by excision. There it sits in a distinguished triangle  $\mathbf{R}\Gamma_x \Omega_{X/k}^n \rightarrow \mathbf{R}\Gamma \Omega_{X/k}^n \rightarrow \mathbf{R}\Gamma_U \Omega_{X/k}^n \rightarrow +1$ , where  $U := \text{Spec } \mathcal{O}_{X,x} - \{x\}$  is the open complement of the closed point  $x$ . Since  $\text{Spec } \mathcal{O}_{X,x}$  is affine,  $\mathbf{R}\Gamma \Omega_{X/k}^n$  is concentrated in degree zero and easy to handle. So to understand  $\mathbf{R}\Gamma_x \Omega_{X/k}^n$  we only really need to understand  $\mathbf{R}\Gamma_U \Omega_{X/k}^n$ . However,  $U$  is covered by  $n$  affine opens  $U_i := \text{Spec } \mathcal{O}_{X,x} - \{(t_i)\}$  where  $t_1, \dots, t_n \in \mathfrak{m}_x$  is any regular sequence. Since these are affine, they also have no higher cohomology. Thus, describing elements in  $H_x^n(X, \Omega_{X/k}^n)$  reduces to providing

- a differential form  $\omega$  in the stalk of  $\Omega_{X/k}^n$  at  $x$ ,
- the regular sequence  $t_1, \dots, t_n$  determining the affine open cover.

One can denote this datum compactly by writing

$$(0.6) \quad \begin{bmatrix} \omega \\ t_1, \dots, t_n \end{bmatrix} \in H_x^n(X, \Omega_{X/k}^n).$$

As this datum describes an element of  $H_x^n(X, \Omega_{X/k}^n)$ , we can evaluate the map of eq. 0.4 on it, denoted  $\text{Res}_{[t_1, \dots, t_n]}^\omega$ . This map is Grothendieck's residue symbol. By the resolution in eq. 0.5 this is the map underlying the co-unit of the duality adjunction. Clearly we have had to make many choices along the way.

Next, one usually checks that if one replaces  $t_1, \dots, t_n$  with a different regular sequence  $t'_1, \dots, t'_n$  where  $t'_i = \sum c_{ij} t_j$ , one has

$$\det(c_{ij}) \cdot \operatorname{Res} \left[ \begin{array}{c} \omega \\ t'_1, \dots, t'_n \end{array} \right] = \operatorname{Res} \left[ \begin{array}{c} \omega \\ t_1, \dots, t_n \end{array} \right]$$

(cf. [11, (R1) on p. 197]). If one has  $\omega = f_0 dt'_1 \cdots dt'_n$ , this of course implies the equation  $\omega = f_0 \det(c_{ij}) dt_1 \cdots dt_n$  so that when writing the differential form in the explicit coordinates of the denominator, as in

$$\operatorname{Res} \left[ \begin{array}{c} f_0 dt_1 \wedge \cdots \wedge dt_n \\ t_1, \dots, t_n \end{array} \right],$$

changing the regular sequence introduces one factor of  $\det(c_{ij})$  and one of  $\det(c_{ij})^{-1}$ , canceling each other out. If  $x \in X$  happens to be a  $k$ -rational point, this means that the map

$$(0.7) \quad \operatorname{Res} \left[ \begin{array}{c} f_0 dt_1 \wedge \cdots \wedge dt_n \\ t_1, \dots, t_n \end{array} \right] := f_0$$

is actually well-defined (cf. [11, (R6) on p. 198]) – but of course this requires a careful proof. It is not hard, but it would be nicer to define the map intrinsically coordinate-independently.

**0.2. Review of the perspective of Tate-Beilinson.** Especially the last equation suggests to read the residue symbol simply as a fraction, i.e. as an entity defined on *rational* differential forms,

$$f_0 \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}.$$

Of course this is an anachronism; this is the classical Cauchy viewpoint. The challenge is to adapt this viewpoint to sheaf theory. Tate and Beilinson achieve this by replacing the Cousin resolution by the so-called adèle resolution. We explain this in §2. For curves this is classical and was imported from number theory via the number field-function field dictionary. Parshin then developed adèles for surfaces [25], and Beilinson handled the general case [1]. See also [23].

Without going into details already here, this flasque resolution has the form

$$(0.8) \quad \Omega_{X/k}^n \simeq \left[ \cdots \rightarrow \text{subspace of } \prod_{\Delta} A(\Delta, \mathcal{O}_X) \otimes \Omega_{X/k}^n \right]_{0,n},$$

where  $\Delta = (\eta_0 > \cdots > \eta_n)$  runs through chains of scheme points so that  $\operatorname{codim}_X \overline{\{\eta_i\}} = i$ . The objects  $A(\Delta, \mathcal{O}_X)$  will be described in detail in §2, they are (finite direct sums of)  $n$ -local fields, i.e. a discrete valuation field whose residue field is a discrete valuation field whose residue field is a discrete... ( $n$  times repeated). Beilinson then defines his residue symbol as a map

$$(0.9) \quad A(\Delta, \mathcal{O}_X) \otimes \Omega_{X/k}^n \longrightarrow k.$$

Any such  $n$ -local field is (non-canonically!) isomorphic to an  $n$ -fold Laurent series field

$$\kappa(x)((t_1))((t_2)) \cdots ((t_n)),$$

where the levelwise uniformizers  $t_1, \dots, t_n$  vaguely correspond to the regular sequence in eq. 0.6, and prime elements along the flag  $\eta_0 > \cdots > \eta_n$ . However, the whole point is of course that Beilinson can construct his symbol in eq. 0.9 *directly*, without the reference to any such isomorphism. This is the main trick to avoid having to make choices.

It remains to construct eq. 0.9. To this end Beilinson uses that the presentation of eq. 0.2 can be generalized to arbitrary dimension. There is a corresponding non-commutative algebra  $E$  of continuous endomorphisms of  $A(\Delta, \mathcal{O}_X)$ . The Lie algebra central extension of

eq. 0.3 corresponds to a Lie cohomology class, which written in Lie homology has the shape  $H_2(E_{Lie}, k) \rightarrow k$ . In the  $n$ -dimensional case Beilinson replaces this with a map

$$(0.10) \quad \phi_{Beil} : H_{n+1}(E_{Lie}, k) \rightarrow k.$$

This does no longer define a central extension of a Lie algebra in a classical sense, but if one presents Lie homology in the classical Chevalley-Eilenberg resolution (this depends only on  $E$ , not on any coordinates), one can define for  $f_0, \dots, f_n \in A(\Delta, \mathcal{O}_X)$ ,

$$f_0 df_1 \wedge \dots \wedge df_n \mapsto f_0 \wedge \dots \wedge f_n \xrightarrow{\phi_{Beil}} k.$$

The composition of these maps yields eq. 0.9. The technical core of the approach then is to construct the Lie homology functional  $\phi_{Beil}$ . This uses a certain cubical object  $T_\bullet$  (in the sense of homological algebra), based on a generalization of ideals of compact and discrete operators in  $E$ ; in dimension one this is already in Tate's paper, but because of the low dimension one can barely recognize the cubical structure there. Neither Tate nor Beilinson explain much about their ideas here – I attempt to give a possible interpretation for  $T_\bullet$  in §4. By homological algebra, there is an isomorphism

$$(0.11) \quad \tilde{\phi} \in \text{Hom}(H_{n+1}(E_{Lie}, k), H_1(T_\bullet)) \cong H^{n+1}(E, H_1(T_\bullet))$$

and one has the so-called ‘‘Tate trace’’  $H_1(T_\bullet) \rightarrow k$ , and composing them Beilinson gets  $\phi_{Beil}$  [1, Lemma (a)]. Doing this explicitly is quite a bit less innocent than eq. 0.11 might suggest. The difficulty of understanding  $\phi_{Beil}$  reduces to understanding the inverse of a differential on the  $(n+1)$ -th page of a certain spectral sequence, namely

$$\phi_{Beil} : H_{n+1}(E_{Lie}, k) \xrightarrow{\text{edge}} E_{0,n+1}^{n+1} \xrightarrow[d-1]{\sim} E_{n+1,1}^{n+1} \xrightarrow{\text{tr} \circ \text{edge}} k.$$

Here ‘edge’ refers to suitable edge maps of the spectral sequence. Even at this level, all can be constructed without ever having to choose coordinates. None of this is new, it can all be found in Beilinson's beautiful two page paper [1], but we add a large amount of supplementary details here.

**0.3. New results.** Next, let me describe what is new in this paper. Let us assume  $\text{char } k = 0$ . For a smooth commutative algebra  $A/k$ , one has the Hochschild-Kostant-Rosenberg (HKR) isomorphism

$$\Omega_{A/k}^n \cong HH_n(A),$$

where  $HH_n(A)$  is Hochschild homology. Hochschild homology admits a Hodge decomposition  $HH_n^{(i)}$  (introduced by Gerstenhaber and Schack [9]) and by smoothness one finds that  $HH_n(A) = HH_n^{(n)}(A)$ , all is concentrated in the Hodge  $n$ -part. The latter looks like the Chevalley-Eilenberg complex computing the Lie homology of  $A$  – as  $A$  is commutative, this Lie algebra structure is of course trivial.

Nonetheless, starting with the HKR isomorphism seems very natural for a theory of residue symbols and the Hodge decomposition seems to be the bridge connecting it with the Lie homology aspects of Beilinson's paper. This simple observation motivates the present paper. In §3 we first recall Beilinson's construction in the language of Lie homology. In §5 we show that it can be lifted to Hochschild homology, we get a ‘‘Hochschild residue symbol’’

$$\phi_{HH} : HH_n(A) \rightarrow k.$$

Due to the aforementioned fact  $HH_n(A) = HH_n^{(n)}(A)$  we do not gain much for smooth commutative algebras  $A$ , but it turns out that the construction works readily for non-commutative algebras.

**Question::** What would Beilinson's residue symbol mean in a non-commutative setup?

Ultimately, we establish a commutative square (up to signs)

$$(0.12) \quad \begin{array}{ccc} H_n(A_{Lie}, A_{Lie}) & \xrightarrow{\varepsilon} & HH_n(A) \\ I' \downarrow & & \downarrow \phi_{HH} \\ H_{n+1}(A_{Lie}, k) & \xrightarrow{\phi_{Beil}} & k, \end{array}$$

where  $A$  can be a non-commutative algebra (this will be Cor. 25). The map  $I'$  denotes a Lie counterpart of the map  $I$  in Connes' periodicity sequence for Hochschild homology. For the use in the adèle resolution, eq. 0.8, it makes no difference to use  $\phi_{Beil}$  or  $\phi_{HH}$ . The map  $\phi_{HH}$  turns out to factor through cyclic homology  $HC_n(A)$ . This is quite nice since if  $A$  is smooth commutative, the HKR isomorphism turns into

$$HC_n^{(n)}(A) \cong \Omega_{A/k}^n / d\Omega_{A/k}^{n-1}$$

( $HC_n^{(n)}$  again denotes the Hodge decomposition). Thinking about residues, it is clear that this should hold. But again,  $\phi_{HH}$  also factors through  $HC_n(A)$  for non-commutative  $A$ .

The main theorem of this paper is an alternative construction of  $\phi_{HH}$  (we call it  $\phi_C$ ). If  $E$  is an associative (possibly non-commutative) algebra and  $I$  a two-sided ideal, so that one gets an algebra extension

$$(0.13) \quad 0 \longrightarrow I \longrightarrow E \longrightarrow E/I \longrightarrow 0,$$

there is a long exact sequence in Hochschild homology

$$(0.14) \quad \cdots \longrightarrow HH_n(E \text{ rel } I) \longrightarrow HH_n(E) \xrightarrow{\iota} HH_n(E/I) \xrightarrow{\delta} HH_{n-1}(E \text{ rel } I) \longrightarrow \cdots$$

The map  $\delta$  turns out to be very interesting: Firstly, we interpret Tate and Beilinson's construction in terms of connecting maps much like  $\delta$ , but in relative Lie homology. This leads to a kind of multi-relative Lie homology, see §4 for details. Concretely, take  $E$  to be the non-commutative algebra of continuous endomorphisms of  $A(\Delta, \mathcal{O}_X)$ , i.e. the same  $E$  as in eq. 0.10, and for  $I$  a suitable ideal of compact and discrete operators, which itself can be interpreted as a variation of  $E$ , but for the divisor  $\overline{\{\eta_1\}}$  cut out by the top entry of the flag  $\eta_1 > \cdots > \eta_n$ . One gets maps

$$HH_n(E) \xrightarrow{\Lambda} HH_n(E/I) \xrightarrow{\delta} HH_{n-1}(E \text{ rel } I) \xrightarrow{\sim} HH_n(I)$$

(of course  $\Lambda \neq \iota$ , otherwise this would just be zero) and iteratively composing such maps, corresponding to going down 'divisor by divisor' in  $\eta_1 > \cdots > \eta_n$ , one gets a map  $HH_n(E) \longrightarrow HH_0(\dots) \longrightarrow k$ . A new feature of this Hochschild version of the theory is the isomorphism  $\sim$  on the right, which allows us to circumvent the aforementioned multi-relative theory, and hereby also all spectral sequences. Our main result is then:

**Theorem 1** (Main Theorem). *These two descriptions of  $\phi_{HH}$  coincide.*

We apologize to the reader that a detailed description of both constructions is a little too long to give it in the introduction, see Thm. 33 for the precise result. Combining this with eq. 0.12 this shows that Beilinson's original residue symbol, at least for input liftable along  $I'$  to  $H_n(A_{Lie}, A_{Lie})$  <sup>(1)</sup> also coincides with this description.

This paper focuses on the local description of the Beilinson-Tate residue symbol. We only briefly discuss Beilinson's theory of adèles since it has already been described in great detail by Huber [13], [14] and Yekutieli [35]. Nor do we say much more about the connection to global duality. It has been sketched above. Detailed discussions have already been given by Hübl,

---

<sup>1</sup>this is for example true if we work with commutative algebras, e.g. in the original application to the residue symbol in algebraic geometry

Sastry and Yekutieli [15], [16, §3], [29]. See also Yekutieli's residue complex [35], [36]. There is also a Hochschild residue theory due to Lipman [17]. Sadly, its relation to the present theory remains unclear to me.

A far more general and elegant, but also technically more demanding approach to the theory than the one explained in the present text proceeds by replacing the ind-pro limits of vector spaces with suitable categories of ind-pro limits over arbitrary exact categories, so-called categories of Tate objects. This is ongoing research joint with Michael Groechenig and Jesse Wolfson. See [4] for category-theoretic foundations, or the work of Previdi [28].

## 1. TATE'S ORIGINAL CONSTRUCTION

**1.1. Operator ideals and the snake lemma.** We shall quickly recall the classical construction of Tate [32], from a perspective which points naturally to the multidimensional generalization. Let  $X/k$  be a smooth algebraic curve. For every closed point  $x \in X$  the completed stalk of the structure sheaf is a complete discrete valuation ring with residue field  $\kappa(x)$ . By Cohen's Structure Theorem there is an isomorphism

$$(1.1) \quad \widehat{\mathcal{K}}_{X,x} := \text{Frac } \widehat{\mathcal{O}}_{X,x} \simeq \kappa(x)((t)),$$

however there is no canonical isomorphism.

*Example 1* (how not to do it). Attempting to construct the residue via a map

$$(1.2) \quad \widehat{\mathcal{K}}_{X,x} \ni \sum a_i t^i \mapsto a_{-1} \in \kappa(x)$$

quickly leads us into trouble. One could use expansions in  $(t + t^2)$  instead of  $t$  for example, or any other isomorphism in eq. 1.1. Even worse, there is no canonical copy of  $\kappa(x)$  inside  $\widehat{\mathcal{K}}_{X,x}$ : Suppose the residue field is  $\mathbf{Q}(s)$  so that  $\widehat{\mathcal{O}}_{X,x} = \mathbf{Q}(s)[[t]]$ . Then  $\mathbf{Q}(s+t) \subset \widehat{\mathcal{O}}_{X,x}$  is a subfield which is mapped isomorphically to  $\mathbf{Q}(s)$  modulo the maximal ideal  $\mathfrak{m}_x = (t)$ . Thus, we get isomorphisms

$$\widehat{\mathcal{K}}_{X,x} \simeq \mathbf{Q}(s)((t)) \quad \text{and} \quad \widehat{\mathcal{K}}_{X,x} \simeq \mathbf{Q}(s+t)((t)),$$

both of which qualify for the residue definition as in eq. 1.2. Hence, both the choice of a coefficient field and the choice of a uniformizing variable  $t$  are non-canonical. Only in exceptional situations one *does* have a canonical coefficient field, notably if the residue field is perfect of positive characteristic or algebraic over the rationals [8, Ch. II §5.2-5.4].

Without needing to choose such an isomorphism,  $\widehat{\mathcal{K}}_{X,x}$  has a canonical topology coming from the presentation  $\widehat{\mathcal{K}}_{X,x} = \varinjlim_s \varprojlim_i \mathcal{O}_{X,x}/\mathfrak{m}_x^i \langle \frac{1}{s} \rangle$ , where we regard each  $\mathcal{O}_{X,x}/\mathfrak{m}_x^i$  as a *discrete*  $k$ -vector space. This turns the inner pro-limit into a linearly compact  $k$ -vector space and the ind-limit over all finitely generated  $\widehat{\mathcal{O}}_{X,x}$ -submodules of  $\widehat{\mathcal{K}}_{X,x}$  into a linearly locally compact  $k$ -vector space.

We can now regard  $\widehat{\mathcal{K}}_{X,x}$  as a infinite-dimensional topological  $k$ -vector space. The topology differs from the ones conventionally used in functional analysis over  $\mathbf{R}$  or  $\mathbf{C}$  because it is generated from an open neighbourhood basis of 0 which consists of linear subspaces; they are called *lattices*:

**Definition 2.** A lattice in a finite-dimensional  $\widehat{\mathcal{K}}_{X,x}$ -vector space  $V$  is a finitely generated  $\widehat{\mathcal{O}}_{X,x}$ -submodule  $L \subseteq V$  so that  $\widehat{\mathcal{K}}_{X,x} \cdot L = V$ .

Using the topology, we get the associative operator algebra of continuous  $k$ -linear endomorphisms

$$(1.3) \quad E := \{ \phi : \widehat{\mathcal{K}}_{X,x} \rightarrow \widehat{\mathcal{K}}_{X,x} \mid \phi \text{ is } k\text{-linear and continuous} \}.$$

**Definition 3.** We call an operator  $\phi \in E$

- (1) compact if there is a lattice  $L$  with  $\text{im } \phi \subseteq L$ ;
- (2) discrete if there is a lattice  $L$  with  $L \subseteq \ker \phi$ .

These classes of operators form two-sided ideals  $I^+, I^-$  in  $E$ . Moreover, we have  $I^+ + I^- = E$ . Write  $I_{tr} := I^+ \cap I^-$  for their intersection. Thus, we get a short exact sequence of  $E$ -bimodules,

$$(1.4) \quad 0 \longrightarrow I_{tr} \longrightarrow I^+ \oplus I^- \longrightarrow E \longrightarrow 0.$$

We may formally “exterior tensor” this with another copy of  $E$ , giving a commutative diagram with exact rows:

$$(1.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & (I^+ \wedge E) \cap (I^- \wedge E) & \longrightarrow & (I^+ \wedge E) \oplus (I^- \wedge E) & \longrightarrow & E \wedge E \longrightarrow 0 \\ & & \downarrow [-, -] & & \downarrow [-, -] & & \downarrow [-, -] \\ 0 & \longrightarrow & I_{tr} & \longrightarrow & I^+ \oplus I^- & \longrightarrow & E \longrightarrow 0 \end{array}$$

(for  $V \subseteq W$  a subspace of a vector space,  $V \wedge W$  denotes the subspace of  $\bigwedge^2 W$  generated by vectors  $v \wedge w$  with  $v \in V, w \in W$ .) The snake lemma gives us a canonical morphism, call it  $(*)$ , and thus

$$(1.6) \quad \phi : \widehat{\mathcal{K}}_{X,x} \wedge \widehat{\mathcal{K}}_{X,x} \longrightarrow \ker(E \wedge E \rightarrow E) \xrightarrow{(*)} \text{coker}([\dots] \rightarrow I_{tr}) \xrightarrow{\text{tr}} k.$$

The local rational functions  $\widehat{\mathcal{K}}_{X,x} \subset E$  are viewed as the respective multiplication operator  $x \mapsto f \cdot x$ , which is clearly continuous. Functions commute, i.e.  $[f, g] = 0$ , so the left-hand side arrow indeed exists. On the other hand, traces satisfy  $\text{tr}([X, Y]) = 0$ , so the trace on the right-hand side factors through the cokernel. Tate now proves that  $\phi(f \wedge g) = \text{res}_x f dg$ . See Lemma 5 for the proof. [32, §2].

*Remark 1.* Tate’s original paper [32] actually defines  $I^+, I^-$  (called  $E_1, E_2$  in *loc. cit.*) slightly differently. He fixes a special open, the ring of integers  $\widehat{\mathcal{O}}_{X,x} \subset \widehat{\mathcal{K}}_{X,x}$ , and instead of compactness he demands an operator to map the entire space into this open, up to a finite-dimensional discrepancy. See also Def. 11. But this comes down to the same as the topological definition we use here. The presentation using a topological language is taken from [2, §1.2] ( $I^+, I^-$  are called  $\text{Hom}_+, \text{Hom}_-$  in *loc. cit.*).

**1.2. Finite-potent trace.** We have tacitly swept a detail under the rug: Since  $E$  is infinite-dimensional, a general operator in  $E$  will not have a well-defined trace. Clearly finite-rank operators will still have a trace, but in Tate’s construction the operators in  $I_{tr}$  a priori need not be of finite rank. In functional analysis one would now hope for the ideal of nuclear operators, but the ind-pro type topologies are not rich enough to give a convergence condition on the operator spectrum any interesting content. Instead, Tate uses the philosophy that any nilpotent operator should have trace zero, even if it is not of finite rank. We briefly summarize Tate’s operator trace [32] as we will also need it later:

Let  $F_0$  be a field and  $V$  an  $F_0$ -vector space. Call an endomorphism  $g \in \text{End}_{F_0}(V)$  *finite-potent* if there is some  $n \geq 1$  such that the image  $g^n V$  is finite-dimensional over  $F_0$ . An  $F_0$ -vector subspace  $\Gamma \subseteq \text{End}_{F_0}(V)$  is called a *finite-potent family* if there is some  $n \geq 1$  such that  $(g_1 \circ \dots \circ g_n)V$  is finite-dimensional for any choice of  $g_1, \dots, g_n \in \Gamma$ .

**Proposition 4** ([32]). *(Tate) For every  $F_0$ -vector space  $V$  and every finite-potent  $g \in \text{End}_{F_0}(V)$  there is a unique element, denoted  $\text{tr}_V g \in F_0$  (and called Tate trace), such that the following rules hold:*

- T1:** If  $V$  is finite-dimensional,  $\text{tr}_V g$  is the usual trace.
- T2:** If  $W \subseteq V$  is any  $F_0$ -vector subspace and  $gW \subseteq W$ , we have  $\text{tr}_V g = \text{tr}_W g + \text{tr}_{V/W} g$ .

**T3:** If  $g$  is nilpotent,  $\text{tr}_V g = 0$ .

**T4:** Suppose  $\Gamma \subseteq \text{End}_{F_0}(V)$  is a finite-potent family. Then  $\text{tr}_V|_\Gamma$  is  $F_0$ -linear, i.e.  $\text{tr}_V(af + bg) = a \text{tr}_V f + b \text{tr}_V g$  for all  $a, b \in F_0$  and  $f, g \in \Gamma$ . <sup>(2)</sup>

**T5:** Suppose  $f : V \rightarrow V'$  and  $g : V' \rightarrow V$  are  $F_0$ -vector space homomorphisms and the composition  $f \circ g$  is finite-potent on  $V'$ . Then the reverse composition  $g \circ f$  is finite-potent on  $V$  and  $\text{tr}_{V'}(f \circ g) = \text{tr}_V(g \circ f)$ .

*Example 2.* Consider  $F_0 := k$  and  $V := k[t, t^{-1}]$ . Then  $f \in \text{End}_{F_0}(V)$  given by  $t^i \mapsto t^{-i}$  for  $i \geq 0$  and  $t^i \mapsto 0$  for  $i < 0$  is a finite-potent morphism which is *not* finite-rank, so the usual trace is not applicable. The vector  $t^0$  spans a 1-dimensional  $f$ -stable subspace and on the vector space quotient  $k[t, t^{-1}]/k\langle t^0 \rangle$  the induced operator  $\bar{f}$  is nilpotent, so by **T1** and **T2** we get  $\text{tr}_V f = 1$ .

**Lemma 5** ([32, Thm. 2]).  $\phi(f \wedge g) = \text{res}_x f dg$ .

*Proof.* We just need to follow the snake morphism in eq. 1.5. For this we need to split the surjection in the top row of eq. 1.5, i.e. idempotents  $P^\pm$  on  $E$  such that  $P^\pm E \subseteq I^\pm$  so that  $P^+ + P^- = 1$ . Then unwinding the snake morphism yields

$$\begin{array}{ccc} (P^+ f \wedge g) \oplus (-P^- f \wedge g) & \longrightarrow & f \wedge g \\ \downarrow & & \\ [P^+ f, g] & \longrightarrow & [P^+ f, g] \oplus -[P^- f, g] \end{array}$$

and so the composition of maps in eq. 1.6 unwinds to the concrete formula

$$(1.7) \quad \phi : \widehat{\mathcal{K}}_{X,x} \wedge \widehat{\mathcal{K}}_{X,x} \rightarrow k \quad \phi(f \wedge g) = \text{tr}[P^+ f, g]$$

(and equivalently  $-\text{tr}[P^- f, g]$  as well). It follows immediately that this formula is independent of the choice of a particular  $P^+$ : Hence, we may pick any isomorphism  $\widehat{\mathcal{K}}_{X,x} \simeq \kappa(x)((t))$ . Suppose  $x$  is a  $k$ -rational point, i.e.  $\kappa(x) = k$ . In order to distinguish between  $t^i$  as a multiplication operator or as a topological basis element of  $\widehat{\mathcal{K}}_{X,x}$ , let us write  $\mathbf{t}^i$  for the latter. Then take for example  $P^+(\mathbf{t}^i) := \delta_{i \geq 0} \mathbf{t}^i$ . This clearly lies in  $I^+$ ,  $P^- := 1 - P^+$  lies in  $I^-$  and we compute

$$[P^+ \mathbf{t}^i, \mathbf{t}^j] \mathbf{t}^\lambda = \delta_{\lambda+i+j \geq 0} \mathbf{t}^{\lambda+i+j} - \delta_{\lambda+i \geq 0} \mathbf{t}^{\lambda+i+j} = \delta_{-j \leq \lambda+i < 0} \mathbf{t}^{\lambda+i+j}.$$

$$(1.8) \quad \begin{array}{c} \mathbf{t}^{-1} \left[ \begin{array}{cccc} \mathbf{t}^0 & \mathbf{t}^1 & \mathbf{t}^2 & \mathbf{t}^3 \end{array} \right] \\ \mathbf{t}^{-1} \mathbf{t}^0 \mathbf{t}^1 \left[ \begin{array}{cc} \mathbf{t}^2 & \mathbf{t}^3 \end{array} \right] \end{array} = \left[ \begin{array}{cc} \mathbf{t}^0 & \mathbf{t}^1 \end{array} \right]$$

Suppose  $j = 1$ , then  $[P^+ \mathbf{t}^i, \mathbf{t}] \mathbf{t}^\lambda = \delta_{-1 \leq \lambda+i < 0} \mathbf{t}^{\lambda+i+1}$ . This has a non-trivial invariant subspace iff  $i = -1$ , so  $\phi(\mathbf{t}^i \wedge \mathbf{t}) = 0$  for  $i \neq -1$ . For  $i = -1$  we get  $[P^+ \mathbf{t}^{-1}, \mathbf{t}] \mathbf{t}^\lambda = \delta_{-1 \leq \lambda-1 < 0} \mathbf{t}^\lambda$ , so  $k\langle \mathbf{t}^0 \rangle$  is a 1-dimensional invariant subspace and therefore  $\phi(\mathbf{t}^{-1} \wedge \mathbf{t}) = 1$ : Just like  $\text{res } t^i dt = \delta_{i=-1}$ . If  $x$  is an arbitrary closed point,  $\kappa(x)/k$  is a finite field extension. The above computation still applies if we work with  $\kappa(x)$ -vector spaces. Writing  $\kappa(x)$  itself as a  $[\kappa(x) : k]$ -dimensional  $k$ -vector space yields the formula  $\text{res } t^i dt = [\kappa(x) : k] \delta_{i=-1}$ .  $\square$

The map  $\phi : \widehat{\mathcal{K}}_{X,x} \wedge \widehat{\mathcal{K}}_{X,x} \rightarrow k$  induces a functional  $H_2((\widehat{\mathcal{K}}_{X,x})_{\text{Lie}}, k)^* \cong H^2((\widehat{\mathcal{K}}_{X,x})_{\text{Lie}}, k)$  and the resulting Lie central extension is the one arising from pushing out eq. 1.4 by Tate's

<sup>2</sup>Mysteriously, in general the linearity axiom **T4** fails. A concrete counter-example is given by Pablos Romo in [24]. However, this need not concern us; the non-linearity will never show up in the applications of the above proposition in this paper.



trace,

$$\begin{array}{ccccccc}
 0 & \longrightarrow & I_{tr} & \longrightarrow & I^+ \oplus I^- & \longrightarrow & E \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & k & \longrightarrow & \widehat{E} & \longrightarrow & E \longrightarrow 0
 \end{array}$$

**Definition 6.** The central extension  $\widehat{E}$  in the lower row is Tate's central extension.

## 2. ADÈLES

**2.1. For curves.** Let  $X/k$  be an integral smooth proper algebraic curve. Tate [32] uses the language of adèles of the curve – a technique borrowed from number theory. We write  $\prod_{x \in U^1} \widehat{\mathcal{O}}_{X,x}$  as a shorthand for the  $\mathcal{O}_X$ -module sheaf

$$U \mapsto \prod_{x \in U^1} \widehat{\mathcal{O}}_{X,x} \quad \text{for } U \text{ any Zariski open set,}$$

where  $\widehat{\mathcal{O}}_{X,x}$  is the  $\mathfrak{m}_x$ -adically completed local ring and  $U^p$  denotes the set of codimension  $p$  points in  $U$ . The restriction map to smaller opens is the factorwise identity so that the sheaf is flasque. There is an exact sequence of  $\mathcal{O}_X$ -module sheaves

$$(2.1) \quad 0 \longrightarrow \mathcal{O}_X \xrightarrow{\text{diag}} k(X) \oplus \prod_{x \in U^1} \widehat{\mathcal{O}}_{X,x} \xrightarrow{\text{diff}} \prod'_{x \in U^1} \widehat{\mathcal{K}}_{X,x} \longrightarrow 0,$$

where  $\mathcal{O}_X$  is the structure sheaf,  $k(X)$  the locally constant sheaf of rational functions,  $\widehat{\mathcal{K}}_{X,x} := \text{Frac } \widehat{\mathcal{O}}_{X,x}$ , and the prime superscript in the rightmost sheaf abbreviates the condition that for all but finitely many  $x \in U^1$  we demand sections to lie in the subspace  $\widehat{\mathcal{O}}_{X,x} \subset \widehat{\mathcal{K}}_{X,x}$ . It is clear that the sequence is exact and that it is actually a flasque resolution of  $\mathcal{O}_X$ . Moreover, the global sections of the sheaves are classically known as

sheaf side	adèle side	
$H^0(X, k(X))$	$k(X)$	function field of the curve
$H^0(X, \prod_{x \in U^1} \widehat{\mathcal{O}}_{X,x})$	$\mathbf{A}_X^0$	integral adèle ring
$H^0(X, \prod'_{x \in U^1} \widehat{\mathcal{K}}_{X,x})$	$\mathbf{A}_X$	adèle ring

The adèle approach to the theory of curves is due to Weil, we refer to [31], [32] for a presentation of this formalism. The same technique works for arbitrary quasi-coherent sheaves by tensoring. As a result of the resolution in eq. 2.1 we obtain for example

$$H^0(X, \mathcal{O}_X) = \mathbf{A}_X^0 \cap k(X) \quad H^1(X, \mathcal{O}_X) = \mathbf{A}_X / (\mathbf{A}_X^0 + k(X)).$$

In particular, in order to describe the global residue map

$$H^1(X, \Omega_{X/k}^1) \longrightarrow k$$

we can employ such an adèle resolution of the sheaf  $\Omega_{X/k}^1$  to give elements of the left-hand side a concrete representation, cf. Tate [32].

**2.2. In general.** Parshin generalized this method to surfaces by introducing 2-dimensional adèles [25], [27]. Beilinson's paper [1] provides the multidimensional technology. We need to recall this for later use:

We mostly follow the notation in [1]. Let  $X$  be a Noetherian scheme. For points  $\eta_0, \eta_1 \in X$  we write  $\eta_0 > \eta_1$  if  $\overline{\{\eta_0\}} \ni \eta_1$ ,  $\eta_1 \neq \eta_0$ . Denote by  $S(X)_n := \{(\eta_0 > \cdots > \eta_n), \eta_i \in X\}$  the set of chains of length  $n+1$ . Let  $K_n \subseteq S(X)_n$  be an arbitrary subset. For any point  $\eta \in X$  define

${}_{\eta}K := \{(\eta_1 > \dots > \eta_n) \text{ s.t. } (\eta_0 > \dots > \eta_n) \in K_n\}$ , a subset of  $S(X)_{n-1}$ . Let  $\mathcal{F}$  be a *coherent* sheaf on  $X$ . For  $n = 0$  and  $n \geq 1$  we define inductively

$$(2.2) \quad \begin{aligned} A(K_0, \mathcal{F}) &:= \prod_{\eta \in K_0} \varprojlim_i \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X, \eta} / \mathfrak{m}_{\eta}^i \\ A(K_n, \mathcal{F}) &:= \prod_{\eta \in X} \varprojlim_i A({}_{\eta}K_n, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X, \eta} / \mathfrak{m}_{\eta}^i). \end{aligned}$$

The sheaves  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X, \eta} / \mathfrak{m}_{\eta}^i$  are usually only quasi-coherent, so we complement this partial definition as follows: For a *quasi-coherent* sheaf  $\mathcal{F}$  we define  $A(K_n, \mathcal{F}) := \varinjlim_{\mathcal{F}_j} A(K_n, \mathcal{F}_j)$ , where  $\mathcal{F}_j$  runs through all coherent subsheaves of  $\mathcal{F}$  (and hereby reducing to eqs. 2.2). As it is built successively from ind-limits and Mittag-Leffler pro-limits,  $A(K_n, -)$  is a covariant exact functor from quasi-coherent sheaves to abelian groups. Next, we observe that  $S(X)_{\bullet}$  carries a natural structure of a simplicial set (omitting the  $i$ -th entry in a flag yields faces; duplicating the  $i$ -th entry in a flag degeneracies). This turns

$$\mathbf{A}^{\bullet}(U, \mathcal{F}) := A(S(U)_{\bullet}, \mathcal{F}) \quad (\text{for } U \text{ Zariski open})$$

into a sheaf of cosimplicial abelian groups and via the unreduced Dold-Kan correspondence into a complex of sheaves, which we may denote by  $\mathbf{A}_{\mathcal{F}}^i$ .

**Theorem 7** ([1, §2]). (*Beilinson*) *For a Noetherian scheme  $X$  and a quasi-coherent sheaf  $\mathcal{F}$  on  $X$ , the  $\mathbf{A}^i(-, \mathcal{F})$  are flasque sheaves and*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathbf{A}_{\mathcal{F}}^0 \longrightarrow \mathbf{A}_{\mathcal{F}}^1 \longrightarrow \dots$$

*is a flasque resolution.*

See Huber [13], [14] for a detailed proof. There are also discussions circling around this construction in Hübl-Yekutieli [16], Osipov [22], Parshin [26]. A very interesting perspective on the relation of the Grothendieck residue complex and adèles can be found in Yekutieli [37]. There should also be a theory of “log-adèles”, possibly in the spirit of Gorchinskiy–Rosly’s polar theory [10]. Beilinson actually defines  $S(X)_n$  so that also degenerate chains with  $\eta_i = \eta_{i+1}$  are allowed, but one can check that this yields a slightly larger, but quasi-isomorphic complex [14, §5.1].

*Example 3.* Suppose  $X$  is an integral smooth proper curve and  $\triangle$  the set of all flags. We may read the sets of codimension  $p$  points  $X^p$  as length one flags. One computes

$$\begin{aligned} A(X^0, \mathcal{O}_X) &= k(X) & A(X^1, \mathcal{O}_X) &= \prod_{x \in X^1} \widehat{\mathcal{O}}_{X, x} \\ A(\triangle, \mathcal{O}_X) &= \prod'_{x \in U^1} \widehat{\mathcal{K}}_{X, x} \end{aligned}$$

so that Thm. 7 reduces to the eq. 2.1.

It is also instructive to have a detailed look at a computation in dimension two:

*Example 4* (generic behaviour). For a commutative and unital ring  $R$  and a prime  $P \subset R$ ,  $\varinjlim_{f \notin P} R[f^{-1}]$  is the localization  $R_P$ . For any such  $f$ ,  $R[f^{-1}] = \varinjlim_i R \langle f^{-i} \rangle$  for  $i \rightarrow \infty$ , where  $\langle f^{-i} \rangle$  denotes the  $R$ -submodule generated by  $f^{-i}$  inside  $R[f^{-1}]$ . Combining both colimits writes  $R_P$  as a colimit of finitely generated  $R$ -modules. We shall abbreviate this colimit by writing  $\varinjlim_{f \notin P} R \langle f^{-\infty} \rangle$ . Now suppose  $X := \operatorname{Spec} k[s, t]$  and  $\triangle := \{(0) > (s) > (s, t)\} \in$

$S(X)_2$  is a singleton set. Then

$$\begin{aligned}
A(\Delta, \mathcal{O}_X) &= A({}_{(0)}\Delta, k(s, t)) = \varinjlim_{f \notin (0)} A({}_{(0)}\Delta, k[s, t] \langle f^{-\infty} \rangle) \\
&= \varinjlim_{f \notin (0)} \varprojlim_i A({}_{(s)}(0)\Delta, k[s, t] \langle f^{-\infty} \rangle \otimes k[s, t]_{(s)} / (s^i)) \\
&= \varinjlim_{f \notin (0)} \varprojlim_i \varinjlim_{g \notin (s)} A({}_{(s)}(0)\Delta, k[s, t] \langle f^{-\infty} \rangle \langle g^{-\infty} \rangle / (s^i)) \\
&= \varinjlim_{f \notin (0)} \varprojlim_i \varinjlim_{g \notin (s)} \varprojlim_j k[s, t]_{(s, t)} \langle f^{-\infty} \rangle \langle g^{-\infty} \rangle / (s^i) / (s, t)^j.
\end{aligned}$$

and this yields

$$\begin{aligned}
&= \varinjlim_{f \notin (0)} \varprojlim_i \varinjlim_{g \notin (s)} k[[s, t]] \langle f^{-\infty} \rangle \langle g^{-\infty} \rangle / (s^i) \\
&= \varinjlim_{f \notin (0)} \varprojlim_i k[[s, t]] [(k[s, t] - (s))^{-1}] \langle f^{-\infty} \rangle / (s^i) \\
&= \varinjlim_{f \notin (0)} k((t))[[s]] \langle f^{-\infty} \rangle = k((t))((s)).
\end{aligned}$$

Note that this computation has not provided us with a canonical isomorphism to  $k((t))((s))$ . Already writing  $\mathbf{A}_k^2$  as  $\text{Spec } k[s, t]$  involved the choice of coordinates  $s, t$ .

The structural similarities to the entire discussion in §1 are more than obvious. Again, we get an isomorphism  $\simeq k((t))((s))$  making it tempting to define a two-dimensional residue as

$$\text{res}_t \text{res}_s f ds \wedge dt = a_{-1, -1} \quad \text{where} \quad f = \sum_{s, t} a_{k, l} s^k t^l.$$

While this would work (cf. [25], [27], but beware of the topological pitfalls explained by Yekutieli [35]) it is a priori again entirely unclear whether this construction is independent of the choice of the isomorphism.

*Example 5* (exceptional behaviour). An example where  $A(\Delta, \mathcal{O}_X)$  has two summands arises at singularities. Note that for  $\text{char } k \neq 2$  the prime ideal  $(s^3 + s^2 - t^2)$  in  $k[s, t]$  does not remain prime under the adelic completion because the new element  $\sqrt{1+s} = \sum_{k \geq 0} \binom{1/2}{k} s^k$  enables a factorization. Instead, we get two irreducible components.



For the flag  $\Delta := \{(0) > (s, t)\}$  we obtain

$$\begin{aligned}
A(\Delta, \mathcal{O}_X) &= A({}_{(0)}\Delta, k(s)[t] / (s^3 + s^2 - t^2)) \\
&= \varinjlim_{f \notin (0)} A({}_{(0)}\Delta, k[s, t] / (s^3 + s^2 - t^2) \langle f^{-\infty} \rangle) \\
&= \varinjlim_{f \notin (0)} \varprojlim_i \varinjlim_{g \notin (s, t)} k[s, t] / (t^i, s^i, s^3 + s^2 - t^2) \langle f^{-\infty} \rangle \langle g^{-\infty} \rangle \\
&= \varinjlim_{f \notin (0)} k[[s, t]] / (s\sqrt{1+s} + t)(s\sqrt{1+s} - t) \langle f^{-\infty} \rangle \\
&= k((s)) \oplus k((s)),
\end{aligned}$$

so that the image of  $t$  is  $(-s\sqrt{1+s}, +s\sqrt{1+s})$ . For the last step in the computation note that the colimit is an Artinian ring, so it is isomorphic to the product over the localizations at its maximal ideals.

A detailed description of the behaviour of adèles especially for flags along singular subvarieties can be found in [26], [27]. One can give a precise dictionary between direct summand decompositions in adèles and fibers of singularities under normalization. We recommend Yekutieli [35, §3.3] for a thorough discussion.

**Definition 8** (see [7]). *For  $n \geq 1$  an  $n$ -local field with last residue field  $k$  is a complete discrete valuation field whose residue field is an  $(n-1)$ -local field with last residue field  $k$ . Moreover, we call  $k$  itself the only 0-local field with last residue field  $k$ .*

**Proposition 9** (Structure theorem, [1, p. 2, 2<sup>nd</sup> paragr.]). *Suppose  $X$  is a finite type reduced scheme of pure dimension  $n$  over a field  $k$  and*

$$\Delta = \{(\eta_0 > \dots > \eta_n) \text{ such that } \text{codim}_X \eta_i = i\} \subseteq S_n(X)$$

*a finite subset. Define*

$$\Delta' := \{(\eta_1 > \dots > \eta_n) \text{ such that } (\eta_0 > \dots > \eta_n) \in \Delta \text{ for some } \eta_0\}.$$

*Then  $A(\Delta, \mathcal{O}_X)$  is a finite direct product of  $n$ -local fields  $\prod K_i$  such that each last residue field is a finite field extension of  $k$ . Moreover,*

$$(2.3) \quad A(\Delta', \mathcal{O}_X) \stackrel{(*)}{\subseteq} \prod \mathcal{O}_i \subseteq \prod K_i = A(\Delta, \mathcal{O}_X),$$

*where  $\mathcal{O}_i$  denotes the ring of integers of  $K_i$  and  $(*)$  is a finite ring extension. Each  $K_i$  is non-canonically isomorphic to  $k'((t_1)) \cdots ((t_n))$  for  $k'/k$  finite. For a coherent sheaf  $\mathcal{F}$ ,  $A(\Delta, \mathcal{F}) \cong \mathcal{F} \otimes_{\mathcal{O}_X} A(\Delta, \mathcal{O}_X)$ .*

*Beware:* Even if  $\Delta$  consists only of one flag, the products in eq. 2.3 may have several factors. See Example 5.

The first published proof of (a mild variation) of the above result was given by Yekutieli [35, Thm. 3.3.2]. We now have described the multidimensional generalization of the infinite-dimensional  $k$ -vector space  $\widehat{\mathcal{K}}_{X,x}$  appearing in §1.

Next, we need to describe the higher analogues of the operator ideals  $I^+, I^-$ . Since these might seem quite involved, let us axiomatize the precise input datum which the following constructions require:

**Definition 10** ([1]). *Let  $k$  be a field. An  $(n$ -fold) cubically decomposed algebra<sup>3</sup> over  $k$  is the datum  $(A, (I_i^\pm), \tau)$ :*

- *an associative  $k$ -algebra  $A$ ;*
- *two-sided ideals  $I_i^+, I_i^-$  such that  $I_i^+ + I_i^- = A$  for  $i = 1, \dots, n$ ;*
- *writing  $I_i^0 := I_i^+ \cap I_i^-$  and  $I_{tr} := I_1^0 \cap \dots \cap I_n^0$ , a  $k$ -linear map (called trace)*

$$\tau : I_{tr} / [I_{tr}, A] \rightarrow k.$$

The essence of Beilinson's residue construction uses nothing but the above datum. The reader should therefore not be discouraged by the involved actual construction of it:

Below  $\text{Hom}_k(-, -)$  refers to plain  $k$ -vector space homomorphisms without any further conditions.

**Definition 11** ([1]). *Suppose  $X/k$  is a finite type reduced scheme of pure dimension  $n$ .*

---

<sup>3</sup>This definition is slightly more general than in [3] because we do not demand that  $A$  is unital.

- (1) Let  $\Delta = \{(\eta_0 > \dots > \eta_i)\}$  be given and  $M$  a finitely generated  $\mathcal{O}_{\eta_0}$ -module. Then a lattice in  $M$  is a finitely generated  $\mathcal{O}_{\eta_1}$ -module  $L \subseteq M$  such that  $\mathcal{O}_{\eta_0} \cdot L = M$ .
- (2) For any quasi-coherent sheaf  $M$  on  $X$  define  $M_\Delta := A(\Delta, M)$ .
- (3) Write  $\Delta' := {}_{\eta_0}\Delta = \{(\eta_1 > \dots > \eta_n)\}$ . Suppose  $M_1, M_2$  are finitely generated  $\mathcal{O}_{\eta_0}$ -modules. Let  $\text{Hom}_\Delta(M_1, M_2)$  be the  $k$ -subalgebra of those  $f \in \text{Hom}_k(M_{1\Delta}, M_{2\Delta})$  such that for all lattices  $L_1 \subset M_1, L_2 \subset M_2$  there exist lattices  $L'_1 \subset M_1, L'_2 \subset M_2$  such that
 
$$(2.4) \quad L'_1 \subseteq L_1, \quad L_2 \subseteq L'_2, \quad f(L'_{1\Delta'}) \subseteq L_{2\Delta'}, \quad f(L_{1\Delta'}) \subseteq L'_{2\Delta'}$$
 and for all such  $L_1, L'_1, L_2, L'_2$  the induced  $k$ -linear map
 
$$(2.5) \quad \bar{f} : (L_1/L'_1)_{\Delta'} \rightarrow (L_2/L'_2)_{\Delta'}$$
 lies in  $\text{Hom}_{\Delta'}(L_1/L'_1, L_2/L'_2)$ . Define  $\text{Hom}_\emptyset(-, -)$  as  $\text{Hom}_k(-, -)$ .
- (4) Define  $I_{1\Delta}^+(M_1, M_2)$  to consist of those  $f \in \text{Hom}_\Delta(M_1, M_2)$  such that there exists a lattice  $L \subset M_2$  with  $f(M_{1\Delta}) \subseteq L_{\Delta'}$ . Respectively,  $I_{1\Delta}^-(M_1, M_2)$  consists of those such that there exists a lattice  $L \subset M_1$  with  $f(L_{\Delta'}) = 0$ . Next, for  $i = 2, \dots, n$  and both  $+/-$  define  $I_{i\Delta}^\pm(M_1, M_2)$  as those  $f \in \text{Hom}_\Delta(M_1, M_2)$  such that for all lattices  $L_1, L'_1, L_2, L'_2$  as in eq. 2.4 we have
 
$$\bar{f} \in I_{(i-1)\Delta'}^\pm(L_1/L'_1, L'_2/L_2).$$

A discussion around this type of structure can be found in Osipov [22]. It can be related to topologizations of  $n$ -local fields [35], [5]. Note the similarity to Definitions 2 and 3. The above involved definition leads us to the central object of study:

**Definition 12** ([1]). In the context of the previous definition, let  $E_\Delta := \text{Hom}_\Delta(\mathcal{O}_{\eta_0}, \mathcal{O}_{\eta_0}) \subseteq \text{End}_k(\mathcal{O}_{X\Delta}, \mathcal{O}_{X\Delta})$ . Write  $I_{i\Delta}^\pm \subseteq E_\Delta$  for  $I_{i\Delta}^\pm(\mathcal{O}_{\eta_0}, \mathcal{O}_{\eta_0})$  and  $i = 1, \dots, n$ .

*Example 6* (toy example [3]). The above definition can easily be confusing. It is helpful to look at the structurally simpler, but essentially equivalent case of infinite matrix algebras first: For any associative algebra  $R$  define

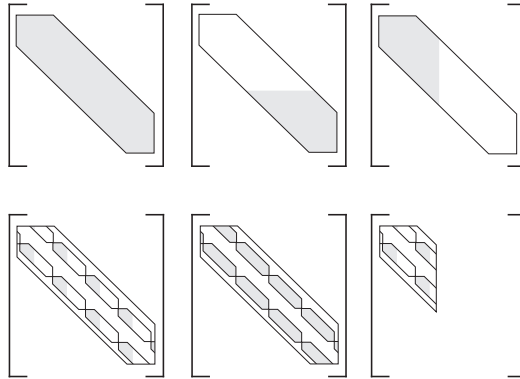
$$(2.6) \quad E(R) := \{\phi = (\phi_{ij})_{i,j \in \mathbf{Z}}, \phi_{ij} \in R \mid \exists K_\phi : |i - j| > K_\phi \Rightarrow \phi_{ij} = 0\}$$

and equip it with the usual matrix multiplication. Then

$$I^+(R) := \{\phi \in E(R) \mid \exists B_\phi : i < B_\phi \Rightarrow \phi_{ij} = 0\}$$

$$I^-(R) := \{\phi \in E(R) \mid \exists B_\phi : j > B_\phi \Rightarrow \phi_{ij} = 0\}$$

define two-sided ideals in  $E(R)$  with  $I^+(R) + I^-(R) = E(R)$ . We may iterate this construction so that  $I_i^\pm := (EE \dots I^\pm \dots E)(R)$  (with  $I^\pm$  in the  $i$ -th place) defines a two-sided ideal of  $E^n(R) = E \dots E(R)$ . One checks that  $(E^n R, \{I_i^\pm\}, \text{tr})$  is an  $n$ -fold cubically decomposed algebra [3, §1.1].



The top row displays typical matrices from  $E(R)$ ,  $I^+(R)$ ,  $I^-(R)$  respectively. The lower row illustrates double infinite matrix constructions, namely  $E(I^-(R))$ ,  $E(E(R))$  and  $I^-(I^-(R))$  respectively. Although defined in a more complicated way, the ideals of Definition 11 have the same structural properties as these infinite matrix ideals. Note that  $E^n(R)$  has a natural  $R$ -linear action on the Laurent polynomial ring  $R[t_1^\pm, \dots, t_n^\pm]$ .

**Proposition 13** ([1, Thm. (a)]). *Suppose  $X/k$  is a finite type reduced scheme of pure dimension  $n$ . Suppose  $\Delta = \{(\eta_0 > \dots > \eta_n)\}$  is a single-element set such that  $\text{codim}_X \eta_i = i$ .*

- (1) *Then  $(E_\Delta, (I_{i\Delta}^\pm), \text{tr}_{I_{tr}})$  is a unital cubically decomposed algebra over  $k$ , where  $\text{tr}_{I_{tr}}$  refers to Tate's operator trace (cf. Prop. 4).*
- (2) *For every  $f \in I_{tr}$  there exists a finite-dimensional  $f$ -stable  $k$ -vector subspace  $W \subseteq E_\Delta$  such that  $\text{tr}_{I_{tr}} f = \text{tr}_W f$ .*

*Proof.* One easily sees that the  $I_i^\pm$  are two-sided ideals. For  $I_i^+ + I_i^- = E_\Delta$  pick any lattice on the suitable level of the inductive definition and any vector space idempotent projecting on it, call it  $P^+$ . Then  $P^- := \mathbf{1} - P^+$  contains the lattice in the kernel. Clearly,  $\mathbf{1} = P^+ + P^-$  and  $P^\pm \in I_i^\pm$ . It remains to check that Tate's trace is defined on  $I_{tr} = I_1^0 \cap \dots \cap I_n^0$ , i.e. that all operators in this ideal are finite-potent, one can argue by induction: Suppose  $f \in I_{tr}(V, V)$  for some  $V$ . In particular  $f \in I_n^0(V, V)$ , i.e. there exists a lattice  $L \subset V$  such that  $fL = 0$  and a lattice  $L' \subset V$  such that  $fV \subseteq L'$ . We observe that  $f^{\circ 3^n} : V \rightarrow V$  factors as

$$(2.7) \quad f^{\circ 3^n} : V \xrightarrow{f^{\circ 3^{n-1}}} L' \xrightarrow{\text{quot}} \frac{L'}{L \cap L'} \xrightarrow{\bar{f}^{\circ 3^{n-1}}} \frac{L'}{L \cap L'} \xrightarrow{f^{\circ 3^{n-1}}} L' \xrightarrow{\text{incl}} V.$$

As  $L, L'$  are lattices,  $L \cap L'$  is a lattice, so we may take  $L'_1 = L_2 := L \cap L'$  and  $L_1 = L'_2 := L'$  as choices in eq. 2.4. As we also have  $f \in I_{n-1}^0$ , this yields that  $\bar{f} \in I_{n-1}^0(L'/(L \cap L'), L'/(L \cap L'))$ . Thus, using  $V := L'/(L \cap L')$  the middle term  $\bar{f}^{\circ 3^{n-1}}$  in eq. 2.7 again satisfies the assumptions for the induction step, just replace  $n$  with  $n-1$ . Proceed down to  $n=1$ , where the middle term  $\bar{f}^{\circ 1}$  is a morphism of finite-dimensional  $k$ -vector spaces. Combining all induction steps, this shows that for every  $f \in I_{tr}$ ,  $f^{\circ 3^n}$  factors through a finite-dimensional  $k$ -vector space  $W$ , so a power of  $f$  indeed has finite-dimensional image over  $k$ , i.e.  $f$  is finite-potent. Similarly, the computation of the trace can be reduced to a classical trace: Again, we use induction. Assume  $f \in I_n^0$ . As the lattices  $L, L'$  (chosen as above) are  $f$ -stable, using axiom **T2** twice yields

$$\text{tr}_V f = \text{tr}_{L'} f + \text{tr}_{V/L'} f = (\text{tr}_{L \cap L'} f + \text{tr}_{L'/L \cap L'} f) + \text{tr}_{V/L'} f.$$

As  $\bar{f} \equiv 0$  in the quotient  $V/L'$  as well as  $f|_L = 0$  when restricted to  $L$  (and thus  $L \cap L'$ ), axiom **T3** reduces the above to  $\text{tr}_{L'/L \cap L'} \bar{f}$ . Hence, we have reduced to  $\bar{f} : L'/(L \cap L') \rightarrow L'/(L \cap L')$ . As before it follows that if we also have  $f \in I_{n-1}^0(V, V)$ , then  $\bar{f} \in I_{n-1}^0(L'/(L \cap L'), L'/(L \cap L'))$  and using  $V := L'/(L \cap L')$  we again satisfy our initial assumptions for the induction step. If  $f \in I_{tr}$ , this inductively yields

$$\text{tr}_V f = \dots = \text{tr}_W \bar{f},$$

where  $W$  is a finite-dimensional  $k$ -vector space. Hence, by **T1** the last trace  $\text{tr}_W \bar{f}$  is the ordinary trace of an endomorphism. For  $f \in [I_{tr}, A]$  use **T5** to see that  $\text{tr}_V f = 0$ .  $\square$

Despite the reduction to a finite rank trace, we remark that it was necessary to use Tate's trace to know a priori that the traces in the construction are well-defined – already before learning that a classical trace would suffice.

### 3. BEILINSON'S CONSTRUCTION

**3.1. Beilinson's functional.** Let us recall Beilinson's construction of the cocycle [1]. We begin with some general considerations:

**Definition 14.** For  $V$  a vector space and  $V' \subseteq V$  a subspace, we define

$$V' \wedge \bigwedge^{r-1} V = \left\{ \begin{array}{l} \text{subspace of } \bigwedge^r V \text{ generated by} \\ v' \wedge v_1 \wedge \cdots \wedge v_{r-1} \text{ with } v' \in V', v_i \in V \end{array} \right\}$$

*Beware:* Note that  $V' \wedge (-)$  is by no means an exact functor in any possible sense. It behaves quite differently from  $V' \otimes (-)$ .

Let  $\mathfrak{g} := A_{Lie}$  be the Lie algebra of an associative algebra  $A$  and  $M$  a  $\mathfrak{g}$ -module. Then one has the Chevalley-Eilenberg complex  $C_i^{Lie}(\mathfrak{g}, M) := M \otimes \bigwedge^i \mathfrak{g}$ , see [20, §10.1.3] for details. Its homology is ordinary Lie homology. We abbreviate  $C_i^{Lie}(\mathfrak{g}) := C_i^{Lie}(\mathfrak{g}, k)$  for trivial coefficients. Let  $\mathfrak{j} \subseteq \mathfrak{g}$  be a Lie ideal. Then the vector spaces

$$(3.1) \quad CE(\mathfrak{j})_r := \mathfrak{j} \wedge \bigwedge^{r-1} \mathfrak{g}$$

for  $r \geq 1$  and  $CE(\mathfrak{j})_0 := k$  define a subcomplex of  $C_r^{Lie}(\mathfrak{g}, k)$  via the identification

$$j \wedge f_1 \wedge \cdots \wedge f_{r-1} \approx 1 \otimes j \wedge f_1 \wedge \cdots \wedge f_{r-1}.$$

The differential turns into the nice expression (cf. [1, first eq.] )

$$(3.2) \quad \delta(f_0 \wedge f_1 \wedge \cdots \wedge f_r) := \sum_{0 \leq i < j \leq r} (-1)^{i+j} [f_i, f_j] \wedge f_0 \wedge \cdots \widehat{f_i} \cdots \widehat{f_j} \cdots \wedge f_r.$$

*Beware:* Due to the difference between  $\mathfrak{j} \wedge (-)$  and  $\mathfrak{j} \otimes (-)$  the homology of  $CE(\mathfrak{j})_\bullet$  does *not* agree with the Lie homology  $H_n(\mathfrak{g}, \mathfrak{j})$  with  $\mathfrak{j}$  seen as a  $\mathfrak{g}$ -module.

Now suppose  $A$  is given the extra structure of a cubically decomposed algebra (cf. Definition 10), i.e.

- two-sided ideals  $I_i^+, I_i^-$  such that  $I_i^+ + I_i^- = A$  for  $i = 1, \dots, n$ ;
- writing  $I_i^0 := I_i^+ \cap I_i^-$  and  $I_{tr} := I_1^0 \cap \cdots \cap I_n^0$ , a  $k$ -linear map

$$\tau : I_{tr}/[I_{tr}, A] \rightarrow k.$$

For any elements  $s_1, \dots, s_n \in \{+, -, 0\}$  we define the *degree*  $\deg(s_1, \dots, s_n) := 1 + \#\{i \mid s_i = 0\}$ . Given the above datum, Beilinson constructs a very interesting family of complexes:

**Definition 15** ([1]). *Define*

$$(3.3) \quad {}^\wedge T_\bullet^p := \bigoplus_{\substack{s_1, \dots, s_n \in \{+, -, 0\} \\ \deg(s_1, \dots, s_n) = p}} \bigcap_{i=1}^n \begin{cases} CE(I_i^+) \bullet & \text{for } s_i = + \\ CE(I_i^-) \bullet & \text{for } s_i = - \\ CE(I_i^+) \bullet \cap CE(I_i^-) \bullet & \text{for } s_i = 0 \end{cases}$$

and  ${}^\wedge T_\bullet^0 := CE(\mathfrak{g})_\bullet$ . View them as complexes in the subscript index  $(-)_\bullet$ .

Each  $CE(I_i^\pm)_\bullet$  is a complex and all their differentials are defined by the same formula, namely eq. 3.2. Thus, the intersection of these complexes has a well-defined differential and is a complex itself. Next, Beilinson shows that

$$(3.4) \quad 0 \longrightarrow {}^\wedge T_\bullet^{n+1} \longrightarrow \cdots \longrightarrow {}^\wedge T_\bullet^1 \longrightarrow {}^\wedge T_\bullet^0 \longrightarrow 0$$

is an exact sequence (now indexed by the superscript) with respect to a suitably defined differential coming from a structure as a cubical object (see [1, §1] or [3, Lemma 4]). Thus, we obtain a bicomplex

$$(3.5) \quad \begin{array}{ccccccc} & & & & \rightarrow \cdots \rightarrow & {}^\wedge T_2^0 & \rightarrow 0 \\ & & & & & \downarrow & \\ & & \text{etc.} & & & \downarrow & \\ 0 \rightarrow & {}^\wedge T_1^{n+1} & \rightarrow & {}^\wedge T_1^n & \rightarrow \cdots \rightarrow & {}^\wedge T_1^0 & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & {}^\wedge T_0^{n+1} & \rightarrow & {}^\wedge T_0^n & \rightarrow \cdots \rightarrow & {}^\wedge T_0^0 & \rightarrow 0. \end{array}$$

Its support is horizontally bounded in degrees  $[n+1, 0]$ , vertically  $(+\infty, 0]$ . As a result, the associated two bicomplex spectral sequences are convergent. Since the rows are exact, the one with  $E^0$ -page differential in direction ‘ $\rightarrow$ ’ vanishes already on the  $E^1$ -page. Thus, this (and therefore both) spectral sequences converge to zero. Now focus on the second spectral sequence, the one with  $E^0$ -page differential in direction ‘ $\downarrow$ ’. Since  $E_{\bullet, \bullet}^{n+2} = 0$  by horizontal concentration in  $[n+1, 0]$ , the differential  $d : E_{n+1, 1}^{n+1} \rightarrow E_{0, n+1}^{n+1}$  on the  $(n+1)$ -st page must be an isomorphism. Upon composing its inverse with suitable edge maps, Beilinson gets a morphism

$$(3.6) \quad \phi_{Beil} : H_{n+1}(\mathfrak{g}, k) \xrightarrow{\sim} H_{n+1}(CE(\mathfrak{g})) \xrightarrow{\text{edge}} E_{0, n+1}^{n+1} \xrightarrow[\sim]{d^{-1}} E_{n+1, 1}^{n+1} \xrightarrow{\text{edge}} H_1(\wedge T_{\bullet}^{n+1}) \xrightarrow{\tau} k.$$

For the left-hand side isomorphism note that  $H_{n+1}(\mathfrak{g}, k) \cong H_{n+1}(CE(\mathfrak{g}))$  just by definition of Lie homology (*Beware:* this is true for  $CE(\mathfrak{j})$  if and only if  $\mathfrak{j} = \mathfrak{g}$ ), and  $\wedge T_{\bullet}^0 := CE(\mathfrak{g})_{\bullet}$  by definition. For the right-hand side map  $\tau$  observe that

$$(3.7) \quad H_1(\wedge T_{\bullet}^{n+1}) = H_1(\bigcap_{i=1}^n \bigcap_{s=\{+, -\}} CE(I_i^s)_{\bullet}) = \frac{\mathfrak{j}}{[\mathfrak{j}, \mathfrak{g}]}$$

for  $\mathfrak{j} := \bigcap_{i=1}^n \bigcap_{s=\{+, -\}} I_i^s = I_{tr}$ . Using the Universal Coefficient Theorem in Lie algebra homology, this is the same as giving an element in  $H^{n+1}(\mathfrak{g}, k) \cong H_{n+1}(\mathfrak{g}, k)^*$ . This is the proof for Beilinson’s result [1, Lemma 1 (a)]. We summarize:

**Proposition 16.** (*Beilinson*) *For every cubically decomposed algebra  $(A, (I_i^{\pm}), \tau)$  and  $\mathfrak{g} := A_{Lie}$  there is a canonical morphism*

$$\phi_{Beil} : H_{n+1}(\mathfrak{g}, k) \longrightarrow k,$$

*or equivalently a canonical Lie cohomology class in  $H^{n+1}(\mathfrak{g}, k)$ .*

Thus, if a commutative  $k$ -algebra  $K$  embeds as  $K \hookrightarrow A$ , we get a morphism

$$\begin{aligned} \text{res} : \Omega_{K/k}^n &\xrightarrow{(\diamond)} H_{n+1}(\mathfrak{g}, k) \xrightarrow{\phi_{Beil}} k \\ f_0 df_1 \wedge \cdots \wedge df_n &\longmapsto f_0 \wedge f_1 \wedge \cdots \wedge f_n \longmapsto \phi_{Beil}(f_0 \wedge \cdots \wedge f_n) \end{aligned}$$

It turns out to be the residue. This is essentially [1, Lemma 1 (b) and Thm. (a)]. For a very explicit proof of this see [3, Thm. 4 and Thm. 5]. Note that  $(\diamond)$  is not really a morphism; it does not respect the relation  $d(xy) = xdy + ydx$ . This washes out after composing with  $\phi_{Beil}$ .

*Remark 2* (reduces to Tate’s theory). It is a general fact from homological algebra that the connecting morphism coming from the snake lemma agrees with the inverse of the suitable differential in the bicomplex spectral sequence applied to the two-row bicomplex which one feeds into the snake lemma. If we apply this remark to eq. 1.5, we readily see how eq. 3.6 transforms into eq. 1.6. This also justifies why  $d^{-1} : E_{0, n+1}^{n+1} \rightarrow E_{n+1, 1}^{n+1}$  is a natural choice to consider.

#### 4. ETIOLOGY (OPTIONAL)

I will try to explain how one could read Tate’s original article and naturally be led to Beilinson’s generalization. Clearly, I am just writing down a possible interpretation here and quite likely it has no connection whatsoever with the actual development of the ideas. Since the original papers [32], [1] say very little about the underlying creative process, this might be of some use. Of course, logically, this section is superfluous.

I would have liked to begin by explaining Cartier’s idea. Tate writes “I arrived at this treatment of residues by considering the special features of the one-dimensional case, after discussing with Mumford an approach of Cartier to Grothendieck’s higher dimensional residue symbol” [32, p. 1]. Pierre Cartier told me that he has never published his approach, it was only



disseminated in seminar talks by Adrien Douady, whom we sadly cannot ask anymore. It seems possible that the original formulation of Cartier's method has fallen into oblivion. So allow me to take Tate's method for granted and proceed to Beilinson's generalization.

Firstly, let us reformulate Tate's original construction. As explained in §1, it begins with an exact sequence of Lie modules

$$(4.1) \quad 0 \longrightarrow I^0 \longrightarrow I^+ \oplus I^- \longrightarrow E \longrightarrow 0.$$

We may read  $I^+ \oplus I^-$  as a Lie algebra itself and hope for  $I^0$  being a Lie ideal in there, so that we could view the sequence as an extension of Lie algebras. However, this fails (e.g.  $[x \oplus x, a \oplus b] = [x, a] \oplus [x, b]$  has no reason to be diagonal). There is an easy remedy, we quotient out

$$(4.2) \quad 0 \longrightarrow I^0 \longrightarrow I^+ \oplus I^- \longrightarrow (I^+ \oplus I^-)/I^0 \longrightarrow 0$$

by  $I^-$  and push out the sequence along this quotient map, giving

$$(4.3) \quad 0 \longrightarrow I^0 \xrightarrow{i} I^+ \xrightarrow{j} I^+/I^0 \longrightarrow 0.$$

Now  $I^0$  is indeed a Lie ideal in  $I^+$  so that this is an extension of Lie algebras. We may take the homology of Lie algebras with trivial coefficients, i.e.  $H_i(-) := H_i(-, k)$ . If  $C_i^{\text{Lie}}(-)$  denotes the underlying Chevalley-Eilenberg complex, we get an obvious induced morphism  $j_* : C_i^{\text{Lie}}(I^+) \rightarrow C_i^{\text{Lie}}(I^+/I^0)$ , which we would like to fit into a long exact sequence. To this end, define *relative Lie homology*  $H_i(I^+ \text{ rel } I^0)$  simply as the co-cone of this morphism  $j_*$ , so that we get a long exact sequence

$$(4.4) \quad \cdots \rightarrow H_{i+1}(I^+/I^0) \xrightarrow{d} H_i(I^+ \text{ rel } I^0) \rightarrow H_i(I^+) \rightarrow H_i(I^+/I^0) \xrightarrow{d} \cdots.$$

*Remark 3.* This is not to be confused with the long exact sequence in Lie homology  $H_i(E, -)$  coming from viewing eq. 4.3 as a short exact sequence of coefficient modules. In eq. 4.4 we change the Lie algebra, not the coefficients.

It would be nice to have a more explicit description of the relative homology groups. Instead of just defining them as an abstract co-cone of complexes, define it (quasi-isomorphically) as the kernel of the map  $j_*$  of Chevalley-Eilenberg complexes. Explicitly, this means that it is the kernel in

$$(4.5) \quad 0 \rightarrow C_i^{\text{Lie}}(I^+ \text{ rel } I^0) \rightarrow \bigwedge^i I^+ \rightarrow \bigwedge^i I^+/I^0 \rightarrow 0.$$

We see that  $C_i^{\text{Lie}}(I^+ \text{ rel } I^0) = I^0 \wedge \bigwedge^{i-1} I^+$ , the subspace spanned by those exterior tensors with at least one slot lying in  $I^0$ ; see Definition 14. Next, let us address the question to compute the connecting homomorphism  $d$  in eq. 4.4. Recall that it is constructed by spelling out the underlying complexes and applying the snake lemma. In the homological degree  $H_2 \xrightarrow{d} H_1$ , this unravels as the snake map of

$$(4.6) \quad \begin{array}{ccccccc} 0 & \longrightarrow & I^0 \wedge I^+ & \longrightarrow & I^+ \wedge I^+ & \longrightarrow & (I^+/I^0) \wedge (I^+/I^0) \longrightarrow 0 \\ & & \downarrow [-, -] & & \downarrow [-, -] & & \downarrow [-, -] \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^+ & \longrightarrow & I^+/I^0 \longrightarrow 0 \end{array}$$

and by comparison with diagram 1.5 we find that the connecting homomorphism

$$(4.7) \quad H_2(I^+/I^0) \longrightarrow H_1(I^+ \text{ rel } I^0)$$

agrees (after precomposing with  $E \cong (I^+ \oplus I^-)/I^0 \twoheadrightarrow I^+/I^0$ ) with the snake map used in Tate's construction, see eq. 1.6. We leave it to the reader to spell this out in detail. In summary: Tate's residue can be read as a connecting homomorphism in relative Lie homology.

In the one-dimensional theory we have the notion of a lattice as in Definition 2, e.g. these are the

$$t^i k[[t]] \subset k((t))$$

for any  $i \in \mathbf{Z}$  – here we temporarily allow ourselves to use explicit coordinates for the sake of exposition. As we proceed to the two-dimensional theory, the analogue of  $k((t))$  will look like  $k((s))((t))$  and we get a more complicated pattern of lattices: First of all, there are the “ $t$ -lattices” like  $t^i k((s))[[t]]$  and the quotient of any two such  $t$ -lattices, say of the pair

$$t^i k((s))[[t]] \subset t^j k((s))[[t]] \quad \text{with} \quad j \leq i,$$

is a finite-dimensional  $k((s))$ -vector space; in this example it is the span

$$\simeq k((s)) \langle t^j, t^{j+1}, \dots, t^{i-1} \rangle.$$

In any such space we now get a notion of an “ $s$ -lattice”, namely just in the previous sense, e.g. if  $i = j + 1$  the quotient is just the span  $\simeq k((s)) \langle t^j \rangle$  and the  $s$ -lattices would be of the shape  $s^i k[[s]] \langle t^j \rangle \subset k((s)) \langle t^j \rangle$  for any  $i \in \mathbf{Z}$ . Two things are important to keep in mind here:

Firstly, for the sake of presentation we have described this in explicit coordinates here. Of course we need to replace the vague notion of “ $t$ -lattices” and “ $s$ -lattices” by something which makes no reference to coordinates. See Definition 11 for Beilinson’s beautiful solution. Secondly, there is a true asymmetry between  $t$  and  $s$ . Note that for a field  $k((s))((t))$  the roles of  $s$  and  $t$  are not interchangeable, unlike for  $k[[s]][[t]]$ . For example,  $\sum_{i \geq 0} s^{-i} t^i$  lies in this field, but  $\sum_{i \geq 0} t^{-i} s^i$  does not describe an actual element of  $k((s))((t))$ . This is why we chose to speak of “ $s$ -lattices” in a *quotient* of  $t$ -lattices, rather than trying to deal with something like  $s^i k[[s]]((t))$ . Note for example that  $\bigcup_{i \in \mathbf{Z}} s^i k[[s]]((t)) \subsetneq k((s))((t))$ . To avoid all pitfalls, it would be best to work in appropriate categories of ind-pro limits right from the start, as in [4].

Based on having two lattice structures instead of just one, in dimension two Beilinson deals with four ideals  $I_1^\pm, I_2^\pm$  instead of just a single pair as in Tate’s construction. We may read the exact sequence in eq. 4.1 as a quasi-isomorphism

$$[I^0 \longrightarrow I^+ \oplus I^-]_{1,0} \xrightarrow{\sim} E$$

with a two-term complex concentrated in homological degrees  $[1, 0]$ . View these ideals as representing the  $t$ -lattices of above (e.g.  $I_1^+$  would be endomorphisms whose image lies in some  $t$ -lattice). Then replicating the analogous structure for  $s$ -lattices leads to the bicomplex

$$\left[ \begin{array}{ccc} I_1^0 \cap I_2^0 & \longrightarrow & I_1^0 \cap I_2^+ \oplus I_1^0 \cap I_2^- \\ \downarrow & & \downarrow \\ I_1^+ \cap I_2^0 \oplus I_1^- \cap I_2^0 & \longrightarrow & I_1^+ \oplus I_1^- \oplus I_2^+ \oplus I_2^- \end{array} \right] \xrightarrow{\sim} E.$$

Accordingly, in the theory for  $n$  dimensions one gets a structure of  $n$  cascading notions of lattices, and correspondingly  $2^n$  ideals  $I_i^\pm$ . The above gets replaced with a quasi-isomorphism to an  $n$ -hypercube. It is a matter of taste whether one prefers to work with multi-complexes or with the ordinary total complex. We prefer the latter, giving a complex concentrated in homological degrees  $[n + 1, 0]$ , see eq. 3.4 and eq. 5.7.

In order to construct the residue map in dimension two, it seems natural to perform the mechanism of dimension one twice, once for each layer of lattices. Hence, one should study the connecting homomorphism analogous to the one in eq. 4.7. However, things get a bit more complicated, because if we try to compose two such connecting homomorphisms, we find that the input of the second step should be the relative Lie homology group which is the output of the first step. This leads to *bi-relative Lie homology*, defined just as the kernel on the left-hand side in

$$0 \rightarrow C_i^{\text{Lie}}(I_1^+ \text{ rel } I_1^0 \text{ rel } I_2^0) \rightarrow C_i^{\text{Lie}}(I_1^+ \text{ rel } I_1^0) \rightarrow C_i^{\text{Lie}}(I_1^+/I_2^0 \text{ rel } I_1^0/I_2^0) \rightarrow 0.$$

Here we allow ourselves to write  $I_1^+/I_2^0$  as a shorthand for  $\frac{I_1^+}{I_2^0 \cap I_1^+}$  to improve legibility. Now we are able to compose the associated connecting homomorphism with the one of eq. 4.7, giving something like

$$H_3(I^+/I_1^0 I_2^0) \xrightarrow{d} H_2(I^+/I_2^0 \text{ rel } I_1^0) \xrightarrow{d} H_1(I^+ \text{ rel } I_1^0 \text{ rel } I_2^0).$$

We should make the bi-relative Lie homology more explicit: Unwinding complexes as in eq. 4.5, we see that

$$0 \rightarrow C_i^{\text{Lie}}(I_1^+ \text{ rel } I_1^0 \text{ rel } I_2^0) \rightarrow I_1^0 \wedge \bigwedge^{i-1} I_1^+ \rightarrow I_1^0/I_2^0 \wedge \bigwedge^{i-1} (I_1^+/I_2^0) \rightarrow 0$$

and therefore

$$(4.8) \quad C_i^{\text{Lie}}(I_1^+ \text{ rel } I_1^0 \text{ rel } I_2^0) = \bigcap_{i=1,2} \left( I_i^0 \wedge \bigwedge^{i-1} I_1^+ \right).$$

The reader will have no difficulty in checking that  $i$ -fold multi-relative Lie homology can be defined accordingly, and leads to further intersections of subcomplexes as in eq. 4.8. This explains the underlying structure of Beilinson's complex  ${}^\wedge T_\bullet^p$ , see eq. 3.3. In fact,  ${}^\wedge T_\bullet^p$  is a tiny bit more complicated because it works with all  $2^n$  ideals  $I_i^\pm$  and  $E$  instead of quotienting out the  $I^-$ -ideals and working with  $I^+$  only, i.e. without the simplification coming from switching from eq. 4.2 to eq. 4.3.

Let us pause for a second. What happens if we ignore Remark 3 and phrase Tate's construction in terms of a long exact sequence, this time with varying coefficients? The diagram 4.6 turns into

$$\begin{array}{ccccccc} 0 & \longrightarrow & I^0 \otimes E & \longrightarrow & I^+ \otimes E & \longrightarrow & (I^+/I^0) \otimes E \longrightarrow 0 \\ & & \downarrow [-,-] & & \downarrow [-,-] & & \downarrow [-,-] \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^+ & \longrightarrow & I^+/I^0 \longrightarrow 0 \end{array}$$

and eq. 4.7 gets replaced by

$$H_1(E, I^+/I^0) \longrightarrow H_0(E, I^0).$$

Besides the index shift, this map also gives Tate's residue.<sup>4</sup> Hence, it is actually possible to set up the entire theory using Lie homology with coefficients instead of relative Lie homology. This is the path taken in the previous paper [3]; the corresponding variant of Beilinson's complex  ${}^\wedge T_\bullet^p$  is called  ${}^\otimes T_\bullet^p$  in *loc. cit.* Both variants in general give different maps (and begin and end in different homology groups), but still they are largely compatible [3, Lemma 6] and both give the multi-dimensional residue [3, Thm. 4 and 5].

The coefficient variant is more manageable for explicit computations: The problem with complexes like  $I^0 \wedge \bigwedge^{i-1} I^+$  is that it is difficult to write down explicit bases for these spaces because the only natural candidate are pure tensors

$$f_0 \otimes f_1 \otimes \cdots \otimes f_{i-1}$$

with  $f_0, \dots, f_{i-1}$  ascendingly taken from an ordered basis of  $I^+$  so that  $f_0 \in I^0$ . Performing calculations, it quickly becomes very tedious to maintain elements in this standard ordered shape.

In the next section §5 we propose yet another point of view. First of all, motivated by the strong relation between the Hodge  $n$ -part of Hochschild homology and Lie homology, we replace Lie homology by (the full) Hochschild homology. This poses no problem since all the

---

<sup>4</sup>I find that this is a nice example of a computation which has at least two homological interpretations. If you get Hochschild on board, this paper shows that there are at least four.

Lie algebras/ideals we have encountered above are actually coming from associative algebras and ordinary ideals. For example, the sequence in eq. 4.4 will be replaced by

$$\cdots \rightarrow HH_{i+1}(I^+/I^0) \xrightarrow{d} HH_i(I^+ \text{ rel } I^0) \rightarrow HH_i(I^+) \rightarrow HH_i(I^+/I^0) \xrightarrow{d} \cdots$$

However, now a substantial simplification occurs: In certain circumstances relative Hochschild homology agrees with absolute Hochschild homology, in the sense that the natural morphism

$$HH_i(I^0) \longrightarrow HH_i(I^+ \text{ rel } I^0)$$

sometimes happens to be an isomorphism. This is known as *excision*; it is easily seen to be wrong for arbitrary ideals but it turns out that the ideals  $I_i^0$  have the necessary property. This spares us from having to work with multi-relative homology at all. Instead, we can just compose the corresponding  $n$  connecting maps, one by one, and we will prove that this again gives the same map, but now its construction necessitates much less effort. We will also see that it is much easier to compute this map explicitly, saving us from a lot of trouble we had to go through in [3].

## 5. HOCHSCHILD AND CYCLIC PICTURE

In this section we will formulate an analogue of Beilinson's construction in the context of Hochschild (and later also cyclic) homology. We follow the natural steps:

- (1) We replace Lie homology with Hochschild homology. This is harmless since cubically decomposed algebras come with an associative product structure anyway. There is a natural map

$$\varepsilon : H_\bullet(A_{Lie}, M_{Lie}) \longrightarrow H_\bullet(A, M)$$

ultimately explaining numerous similarities.

- (2) The Hochschild complex is modelled on chain groups  $A \otimes \cdots \otimes A$  instead of exterior powers. Thus, the only reasonable replacement of the mixed exterior powers/relative homology groups

$$CE(j)_r := j \wedge \bigwedge^{r-1} \mathfrak{g}$$

in the original construction are the groups  $J \otimes A \otimes \cdots \otimes A$  for  $J$  an ideal. This is very convenient, as this just gives Hochschild homology with coefficients  $H_r(A, J)$ . We will return to a relative perspective in §6.

To set up notation, let us very briefly recall the necessary structures in Hochschild homology. See [20, Ch. I] for a detailed treatment. Suppose  $A$  is an arbitrary (not necessarily unital) associative  $k$ -algebra. Let  $M$  be an  $A$ -bimodule over  $k$ , or equivalently a left- $A \otimes_k A^{\text{op}}$ -module. Define chain groups  $C_i(A, M) := M \otimes_k A^{\otimes i}$  and a differential  $b : C_i(A, M) \rightarrow C_{i-1}(A, M)$ , given by

$$\begin{aligned} m \otimes a_1 \otimes \cdots \otimes a_i &\mapsto ma_1 \otimes a_2 \otimes \cdots \otimes a_i \\ (5.1) \quad &+ \sum_{j=1}^{i-1} (-1)^j m \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_i \\ &+ (-1)^i a_i m \otimes a_1 \otimes \cdots \otimes a_{i-1}. \end{aligned}$$

We call the homology of the complex  $(C_\bullet(A, M), b)$  its *Hochschild homology*; it is denoted by  $H_i(A, M)$ . Write  $A_{Lie}$  for the Lie algebra associated to  $A$  via  $[x, y] := x \cdot y - y \cdot x$ . There is a canonical morphism

$$\begin{aligned} (5.2) \quad \varepsilon : C_i^{\text{Lie}}(A_{Lie}, M_{Lie}) &\rightarrow C_i(A, M) \\ m \otimes a_1 \wedge \cdots \wedge a_i &\mapsto m \otimes \sum_{\pi \in \mathfrak{S}_i} \text{sgn}(\pi) a_{\pi^{-1}(1)} \otimes \cdots \otimes a_{\pi^{-1}(i)}, \end{aligned}$$

where  $\mathfrak{S}_i$  is the symmetric group on  $i$  letters. This is a morphism of complexes, in particular it induces a morphism  $H_i(A_{Lie}, M_{Lie}) \rightarrow H_i(A, M)$ .

For the rest of this section assume  $A$  is *unital*. Clearly  $A$  is a bimodule over itself and we write  $HH_i(A) := H_i(A, A)$  as an abbreviation (see §6.2 for the correct definition when  $A$  is not unital). A  $k$ -algebra morphism  $f : A \rightarrow A'$  induces a map  $f_* : HH_i(A) \rightarrow HH_i(A')$ . The motivation for using Hochschild homology in the context of residue theory stems from the following famous isomorphism:

**Proposition 17.** (*Hochschild-Kostant-Rosenberg*) *Suppose  $A/k$  is a commutative smooth  $k$ -algebra. Then the morphism*

$$(5.3) \quad \begin{aligned} \Omega_{A/k}^n &\longrightarrow HH_n(A) \\ f_0 df_1 \wedge \cdots \wedge df_n &\longmapsto \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) f_0 \otimes f_{\pi^{-1}(1)} \otimes \cdots \otimes f_{\pi^{-1}(n)} \end{aligned}$$

*is an isomorphism of graded commutative algebras.*

See [20, Thm. 3.4.4]. Let us now assume that  $\mathbf{Q} \subseteq k$ : On  $A^{\otimes(i+1)}$  recall that there is an action by Connes' cyclic permutation operator

$$t : a_0 \otimes a_1 \otimes \cdots \otimes a_i \mapsto (-1)^i a_i \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1}.$$

Define the cyclic chain groups by  $CC_i(A) := A^{\otimes(i+1)}/(1-t)$ ; this is the quotient by the action of  $t$  on pure tensors. As was discovered by Connes, it turns out that the differential  $b$  remains well-defined on these quotients. Its homology is known as *cyclic homology* and denoted by  $HC_i(A)$ . We shall also need *Connes' periodicity sequence* [20, Thm. 2.2.1]: There is a long exact sequence

$$(5.4) \quad \cdots \longrightarrow HH_i(A) \xrightarrow{I} HC_i(A) \xrightarrow{S} HC_{i-2}(A) \xrightarrow{B} HH_{i-1}(A) \longrightarrow \cdots$$

where  $I$  is induced from the obvious inclusion/quotient map on the level of complexes.

*Remark 4.* At the expense of a more complicated definition of the cyclic chain groups, all of these facts remain available without the simplifying assumption  $\mathbf{Q} \subset k$ ; see [20, Thm. 2.1.5, we work with  $H^\lambda$  of *loc. cit.*]. We leave the necessary modifications to the reader.

We shall also need the map (recall that  $\mathfrak{g} := A_{Lie}$ )

$$(5.5) \quad \begin{aligned} I' : H_n(\mathfrak{g}, \mathfrak{g}) &\longrightarrow H_{n+1}(\mathfrak{g}, k) \\ f_0 \otimes f_1 \wedge \cdots \wedge f_n &\longmapsto (-1)^n \otimes f_0 \wedge \cdots \wedge f_n \end{aligned}$$

in Lie homology. The  $(-1)^n$  is needed to make the differentials compatible.

**Proposition 18.** (*Connes, Loday-Quillen*) *Suppose  $A/k$  is a commutative smooth  $k$ -algebra and  $\text{char } k = 0$ . Then there is a canonical isomorphism*

$$HC_n(A) \rightarrow \Omega_{A/k}^n / d\Omega_{A/k}^{n-1} \oplus \bigoplus_{i \geq 1} H_{\text{dR}}^{n-2i}(A)$$

*so that  $I : HH_n(A) \rightarrow HC_n(A)$  identifies with the quotient map  $\Omega_{A/k}^n \rightarrow \Omega_{A/k}^n / d\Omega_{A/k}^{n-1}$  and zero on the lower deRham summands.*

See [20, Thm. 3.4.12 and remark]. The direct summand decomposition on the right-hand side can be identified with the Hodge decomposition of cyclic homology due to Gerstenhaber and Schack [9].

**5.1. Hochschild setup.** Let  $A$  be a cubically decomposed algebra over  $k$ . We define  $A$ -bimodules  $N^0 := A$  and for  $p \geq 1$

$$(5.6) \quad N^p := \bigoplus_{\substack{s_1, \dots, s_n \in \{+, -, 0\} \\ \deg(s_1, \dots, s_n) = p}} I_1^{s_1} \cap I_2^{s_2} \cap \dots \cap I_n^{s_n}$$

with degree  $\deg(s_1, \dots, s_n) := 1 + \#\{i \mid s_i = 0\}$  as before. Each  $I_i^\pm$  is a two-sided ideal and thus an  $A$ -bimodule.

We shall denote the components  $f = (f_{s_1 \dots s_n})$  of elements in  $N^p$  with indices in terms of  $s_1, \dots, s_n \in \{+, -, 0\}$ . Clearly  $N^p = 0$  for  $p > n + 1$ . We get an exact sequence of  $A$ -bimodules

$$(5.7) \quad 0 \longrightarrow N^{n+1} \xrightarrow{\partial} N^n \xrightarrow{\partial} \dots \xrightarrow{\partial} N^0 \longrightarrow 0$$

by using the following differential

$$\begin{aligned} (\partial f)_{s_1 \dots s_n} &:= \sum_{\{i \mid s_i = +, -\}} (-1)^{\#\{j \mid j > i \text{ and } s_j = 0\}} f_{s_1 \dots 0 \dots s_n} \quad (\text{for } N^i \rightarrow N^{i-1}, i \geq 2) \\ \partial f &:= \sum_{s_1 \dots s_n \in \{+, -\}} (-1)^{s_1 + \dots + s_n} f_{s_1 \dots s_n} \quad (\text{for } N^1 \rightarrow N^0) \end{aligned}$$

It is straight-forward to check that  $\partial^2 = 0$  holds, but more details are found in [3, §4] nonetheless. As tensoring with  $(-) \otimes A^{\otimes(r-1)}$  is exact, we can functorially take the Hochschild complex and obtain a bicomplex with exact rows, fairly similar to the bicomplex that we have encountered before in eq. 3.5,

$$(5.8) \quad \begin{array}{ccccccc} & & & & \rightarrow \dots \rightarrow & C_2(A, N^0) & \rightarrow 0 \\ & & \text{etc.} & & & \downarrow & \\ 0 \rightarrow & C_1(A, N^{n+1}) & \rightarrow & C_1(A, N^n) & \rightarrow \dots \rightarrow & C_1(A, N^0) & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & C_0(A, N^{n+1}) & \rightarrow & C_0(A, N^n) & \rightarrow \dots \rightarrow & C_0(A, N^0) & \rightarrow 0 \end{array}$$

As before its support is horizontally bounded in degrees  $[n+1, 0]$ , vertically  $(+\infty, 0]$ ; we get an analogous differential on the  $E^{n+1}$ -page, which is an isomorphism. Proceeding as before, *but this time considering degree  $n$  instead of  $n+1$* , we obtain

$$(5.9) \quad \phi_{HH} : HH_n(A) \xrightarrow{\sim} H_n(A, N^0) \xrightarrow{\text{edge}} E_{0,n}^{n+1} \xrightarrow[d-1]{\sim} E_{n+1,0}^{n+1} \xrightarrow{\text{edge}} H_0(A, N^{n+1}) \xrightarrow{\tau} k.$$

The consideration with the trace  $\tau$  of the cubically decomposed algebra is exactly the same as before since

$$H_0(A, N^{n+1}) = \frac{N^{n+1}}{[N^{n+1}, A]},$$

but  $N^{n+1} = I_1^{s_1} \cap I_2^{s_2} \cap \dots \cap I_n^{s_n} = I_{tr}$ , so we obtain exactly the same object as in the Lie counterpart, see eq. 3.7. In particular, the trace  $\tau$  is applicable for the same reasons as before. This leads to the following new construction:

**Proposition 19.** *For every cubically decomposed algebra  $(A, (I_i^\pm), \tau)$  over  $k$  there is a canonical morphism*

$$\phi_{HH} : HH_n(A) \longrightarrow k.$$

Let us explain how to obtain an explicit formula for the fairly abstract construction of  $\phi_{HH}$ . To this end, we employ the following tool from the theory of spectral sequences:

**Lemma 20** ([3, Lemma 5]). *Suppose we are given a bounded exact sequence*

$$S^\bullet = [S^{n+1} \rightarrow S^n \rightarrow \dots \rightarrow S^0]_{n+1,0}$$

*of bounded below complexes of  $k$ -vector spaces; or equivalently a correspondingly bounded bicomplex.*

- (1) *There is a second quadrant homological spectral sequence  $(E_{p,q}^r, d_r)$  converging to zero such that*

$$E_{p,q}^1 = H_q(S_\bullet^p). \quad (d_r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r)$$

- (2) *The following differentials are isomorphisms:*

$$d_{n+1} : E_{n+1,0}^{n+1} \rightarrow E_{0,n}^{n+1}.$$

- (3) *If  $H_p : S^p \rightarrow S^{p+1}$  is a contracting homotopy for  $S^\bullet$ , then*

$$(5.10) \quad (d_{n+1})^{-1} = H_n \delta_1 H_{n-1} \cdots \delta_{n-1} H_1 \delta_n H_0 = H_n \prod_{i=1,\dots,n} (\delta_i H_{n-i}).$$

This result can be applied to the bicomplex of eq. 5.8. The required contracting homotopy can be constructed from a suitable family of commuting idempotents in the cubically decomposed algebra:

**Definition 21** ([3, Def. 4]). *Suppose  $A$  is unital. A system of good idempotents are pairwise commuting elements  $P_i^+ \in A$  (with  $i = 1, \dots, n$ ) such that the following conditions are met:*

- $P_i^{+2} = P_i^+$ .
- $P_i^+ A \subseteq I_i^+$ .
- $P_i^- A \subseteq I_i^-$  (and we define  $P_i^- := \mathbf{1}_A - P_i^+$ ).

Then the elements  $P_i^-$  are pairwise commuting idempotents as well. We can use the contracting homotopy developed in an earlier paper:

**Lemma 22** ([3, §4, Lem. 3]). *Let  $A$  be unital and  $\{P_i^+\}$  a system of good idempotents. An explicit contracting homotopy  $H : N^i \rightarrow N^{i+1}$  for the complex  $N^\bullet$  of eq. 5.7 is given by*

$$(5.11) \quad (Hf)_{s_1 \dots s_n} = (-1)^{\deg(s_1 \dots s_n)} (-1)^{s_1 + \dots + s_b} P_1^{s_1} \dots P_b^{s_b} \sum_{\gamma_1 \dots \gamma_{b+1} \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_b} P_{b+1}^{-\gamma_{b+1}} f_{\gamma_1 \dots \gamma_{b+1} s_{b+2} \dots s_n}$$

(for  $N^i \rightarrow N^{i+1}$  with  $i \geq 1$ )

$$(5.12) \quad (Hf)_{s_1 \dots s_n} := (-1)^{s_1 + \dots + s_n} P_1^{s_1} \dots P_n^{s_n} f \quad (\text{for } N^0 \rightarrow N^1)$$

where  $b$  is the largest index such that  $s_1, \dots, s_b \in \{\pm\}$  or  $b = 0$  if none.

By tensoring  $(-) \otimes A^{\otimes(r-1)}$  this induces a contracting homotopy for the rows in the bicomplex of eq. 5.8. The evaluation of the formula in eq. 5.10 corresponds to following a zig-zag in the bicomplex which can be depicted graphically as:

$$(5.13) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & | & & \\ & & & \theta_{1,n} & \xleftarrow{H} & \theta_{0,n} & n \\ & & & \vdots & & & \vdots \\ & & \theta_{n,1} & \xleftarrow{H} & \theta_{n-1,1} & & 1 \\ & & \downarrow & & & & \\ \theta_{n+1,0} & \xleftarrow{H} & \theta_{n,0} & & & & 0 \\ n+1 & & n & & n-1 & \dots & 0 \end{array}$$

If  $\theta_{0,n} = f_0 \otimes \dots \otimes f_n$  represents an element in  $E_{0,n}^{n+1}$  arising from the first part of the definition of  $\phi_{HH}$  (cf. eq. 5.9)

$$HH_n(A) \xrightarrow{\sim} H_n(A, N^0) \xrightarrow{\text{edge}} E_{0,n}^{n+1} \ni \theta_{0,n},$$

we can compute  $d^{-1} : E_{0,n}^{n+1} \xrightarrow{\sim} E_{n+1,0}^{n+1}$  by eq. 5.10. We claim:

**Lemma 23.** *Let  $A$  be unital and  $\{P_i^+\}$  a system of good idempotents. Starting with  $\theta_{0,n} = f_0 \otimes \cdots \otimes f_n$ , we get for  $s_1, \dots, s_{n-p} \in \{+, -\}$  the formula*

$$\begin{aligned} \theta_{p+1,n-p|s_1\dots s_{n-p}0\dots 0} &= (-1)^{n+(n-1)+\dots+(n-p+1)} (-1)^{2+3+\dots+(p+1)} (-1)^{s_1+\dots+s_{n-p}} P_1^{s_1} \dots P_{n-p}^{s_{n-p}} \\ &\quad \dots \left( \sum_{\gamma_{n-p+1} \in \{\pm\}} (-1)^{\gamma_{n-p+1}} P_{n-p+1}^{-\gamma_{n-p+1}} f_{n-p+1} P_{n-p+1}^{\gamma_{n-p+1}} \right) \cdot (\dots) \cdot \\ &\quad \dots \left( \sum_{\gamma_n \in \{\pm\}} (-1)^{\gamma_n} P_n^{-\gamma_n} f_n P_n^{\gamma_n} \right) f_0 \otimes f_1 \otimes \cdots \otimes f_{n-p} \end{aligned}$$

for the terms in Fig. 5.13.

This is the Hochschild counterpart of [3, Prop. 1]. The proof will be very similar to the one given for the Lie homology counterpart in [3], but actually quite a bit *less* involved.

*Proof.* We prove this by induction on  $p$ , starting from  $p = 0$ . In this particular case, the claim reads

$$\theta_{1,n|s_1\dots s_n} = (-1)^{s_1+\dots+s_n} P_1^{s_1} \dots P_n^{s_n} f_0 \otimes f_1 \otimes \cdots \otimes f_n,$$

which is clearly true in view of eq. 5.12. Next, assume the claim is known for a given  $p$  and we want to treat the case  $p+1$ , i.e. we need to evaluate a Hochschild differential  $b$  and pick a preimage as in the step

$$\begin{array}{ccc} & \theta_{p+1,n-p} & \\ & \downarrow b & \\ \theta_{p+2,n-p-1} & \xleftarrow{H} & \theta_{p+1,n-p-1} \end{array}$$

of Fig. 5.13. According to our induction hypothesis, we get  $\theta_{p+1,n-p|s_1\dots s_{n-p}0\dots 0} = M f_0 \otimes f_1 \otimes \cdots \otimes f_{n-p}$  with the auxiliary expression

$$\begin{aligned} M &= (-1)^{n+(n-1)+\dots+(n-p+1)} (-1)^{2+3+\dots+(p+1)} (-1)^{s_1+\dots+s_{n-p}} P_1^{s_1} \dots P_{n-p}^{s_{n-p}} \\ &\quad \dots \left( \sum_{\gamma_{n-p+1} \in \{\pm\}} (-1)^{\gamma_{n-p+1}} P_{n-p+1}^{-\gamma_{n-p+1}} f_{n-p+1} P_{n-p+1}^{\gamma_{n-p+1}} \right) \cdot (\dots) \cdot \\ &\quad \dots \left( \sum_{\gamma_n \in \{\pm\}} (-1)^{\gamma_n} P_n^{-\gamma_n} f_n P_n^{\gamma_n} \right). \end{aligned}$$

The Hochschild differential  $b$  naturally decomposes into three parts (cf. eq. 5.1)

$$\begin{aligned} \theta_{p+1,n-p-1}^{(A)} &= M f_0 f_1 \otimes f_2 \otimes \cdots \otimes f_{n-p}, \\ \theta_{p+1,n-p-1}^{(B)} &= \sum_{j=1}^{n-p-1} (-1)^j M f_0 \otimes f_1 \otimes \cdots \otimes f_j f_{j+1} \otimes \cdots \otimes f_{n-p}, \\ \theta_{p+1,n-p-1}^{(C)} &= (-1)^{n-p} f_{n-p} M f_0 \otimes f_1 \otimes \cdots \otimes f_{n-p-1} \end{aligned}$$

(here we have suppressed the subscript  $(-)\_{s_1\dots s_{n-p}0\dots 0}$  for the sake of readability). Next, we need to evaluate  $\theta_{p+2,n-p-1}^{(-)} := H\theta_{p+1,n-p-1}^{(-)}$  for the cases  $A, B, C$ . Let us consider case  $C$ : In



this case, we just use eq. 5.11 and plugging in  $M$ ,

$$\begin{aligned} \theta_{p+2,n-p-1|s_1\dots s_{n-p-1}0\dots 0}^{(C)} &= (-1)^{n-p} (-1)^{\deg(s_1\dots s_{n-p-1}0\dots 0)} (-1)^{s_1+\dots+s_{n-p-1}} P_1^{s_1} \dots P_{n-p-1}^{s_{n-p-1}} \\ &\quad \sum_{\gamma_1\dots\gamma_{n-p}\in\{\pm\}} (-1)^{\gamma_1+\dots+\gamma_{n-p-1}} P_{n-p}^{-\gamma_{n-p}} f_{n-p} \\ &\quad (-1)^{n+(n-1)+\dots+(n-p+1)} (-1)^{2+3+\dots+(p+1)} (-1)^{\gamma_1+\dots+\gamma_{n-p}} P_1^{\gamma_1} \dots P_{n-p}^{\gamma_{n-p}} \\ &\quad \dots \left( \sum_{\gamma_{n-p+1}\in\{\pm\}} (-1)^{\gamma_{n-p+1}} P_{n-p+1}^{-\gamma_{n-p+1}} f_{n-p+1} P_{n-p+1}^{\gamma_{n-p+1}} \right) \cdot (\dots) \cdot \\ &\quad \dots \left( \sum_{\gamma_n\in\{\pm\}} (-1)^{\gamma_n} P_n^{-\gamma_n} f_n P_n^{\gamma_n} \right) f_0 \otimes f_1 \otimes \dots \otimes f_{n-p-1} \end{aligned}$$

This fairly complicated expression unwinds into something much simpler by several observations: (1) There is a large cancellation in the sign terms  $(-1)^{(\dots)}$ , (2) we have  $\deg(s_1\dots s_{n-p-1}0\dots 0) = p+2$ , (3) the pairwise commutativity of the idempotents allows us to reorder terms so that we obtain the expression  $\sum_{\gamma_1\dots\gamma_{n-p-1}\in\{\pm\}} P_1^{\gamma_1} \dots P_{n-p-1}^{\gamma_{n-p-1}}$ , but this is just the identity operator by using the fact  $P_i^+ + P_i^- = \mathbf{1}$ . Finally, we arrive at

$$\begin{aligned} \theta_{p+2,n-p-1|s_1\dots s_{n-p-1}0\dots 0}^{(C)} &= (-1)^{n+(n-1)+\dots+(n-p+1)+(n-p)} (-1)^{2+3+\dots+(p+2)} \\ &\quad (-1)^{s_1+\dots+s_{n-p-1}} P_1^{s_1} \dots P_{n-p-1}^{s_{n-p-1}} \\ &\quad \left( \sum_{\gamma_{n-p}\in\{\pm\}} (-1)^{\gamma_{n-p}} P_{n-p}^{-\gamma_{n-p}} f_{n-p} P_{n-p}^{\gamma_{n-p}} \right) \dots \\ &\quad \dots \left( \sum_{\gamma_n\in\{\pm\}} (-1)^{\gamma_n} P_n^{-\gamma_n} f_n P_n^{\gamma_n} \right) f_0 \otimes f_1 \otimes \dots \otimes f_{n-p-1}. \end{aligned}$$

In a similar fashion we can deal with the cases  $A, B$ , however in both these cases we obtain a term  $P_i^{\gamma_i} P_i^{-\gamma_i} = P_i^{\gamma_i} (\mathbf{1} - P_i^{\gamma_i}) = 0$ , so that these terms vanish. We leave the details to the reader (a similar cancellation occurs in the proof of [3, Prop. 1], the cancellation is explained by the very beautiful identity<sup>5</sup>  $H^2 = 0$ , which holds for this particular contracting homotopy). Hence,  $\theta_{p+2,n-p-1} = \theta_{p+2,n-p-1}^{(C)}$ , giving the claim.  $\square$

**Theorem 24.** *Let  $(A, (I_i^\pm), \tau)$  be a unital cubically decomposed algebra over  $k$  and  $\{P_i^+\}$  a system of good idempotents. Then the explicit formula*

$$\begin{aligned} \phi_{HH}(f_0 \otimes \dots \otimes f_n) &= (-1)^n \tau \left( \sum_{\gamma_1\in\{\pm\}} (-1)^{\gamma_1} P_1^{-\gamma_1} f_1 P_1^{\gamma_1} \right) \dots \\ &\quad \dots \left( \sum_{\gamma_n\in\{\pm\}} (-1)^{\gamma_n} P_n^{-\gamma_n} f_n P_n^{\gamma_n} \right) f_0 \end{aligned}$$

holds.

*Proof.* Use the lemma with  $p = n$  and compose with the trace  $\tau$  as in the definition of  $\phi_{HH}$  in eq. 5.9.  $\square$

---

<sup>5</sup>pointed out to me by the anonymous referee of [3]

**Corollary 25.** *Let  $(A, (I_i^\pm), \tau)$  be a unital cubically decomposed algebra over  $k$ , and let  $\mathfrak{g} := A_{Lie}$  be the associated Lie algebra. The diagram*

$$\begin{array}{ccc} H_n(\mathfrak{g}, \mathfrak{g}) & \xrightarrow{\varepsilon} & HH_n(A) \\ I' \downarrow & & \downarrow \phi_{HH} \\ H_{n+1}(\mathfrak{g}, k) & \xrightarrow{\phi_{Beil}} & k \end{array}$$

*commutes up to sign. Here  $\varepsilon$  refers to the comparison map from eq. 5.2. The composition  $\phi_{HH} \circ \varepsilon$  is given by the commutator formula*

$$(5.14) \quad f_0 \otimes f_1 \wedge \cdots \wedge f_n \mapsto (-1)^n \tau \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \sum_{\gamma_1 \dots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_n} (P_1^{-\gamma_1} \text{ad}(f_{\sigma^{-1}(1)}) P_1^{\gamma_1}) \cdots (P_n^{-\gamma_n} \text{ad}(f_{\sigma^{-1}(n)}) P_n^{\gamma_n}) f_0.$$

*If  $n = 1$  and  $[f_0, f_1] = 0$  then this specializes to*

$$(5.15) \quad f_0 \otimes f_1 \mapsto -\tau[P_1^+ f_0, f_1].$$

The last equation links these formulae with the classical one-dimensional case as found in eq. 1.7.

*Proof.* Let  $\{P_i^+\}$  be any system of good idempotents. A direct computation of  $\phi_{HH} \circ \varepsilon$  yields the explicit formula

$$\begin{aligned} f_0 \otimes f_1 \wedge \cdots \wedge f_n \mapsto & (-1)^n \tau \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \sum_{\gamma_1 \dots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_n} \\ & (P_1^{-\gamma_1} f_{\sigma^{-1}(1)} P_1^{\gamma_1}) \cdots (P_n^{-\gamma_n} f_{\sigma^{-1}(n)} P_n^{\gamma_n}) f_0, \end{aligned}$$

which agrees (up to sign) with the morphism  $\otimes \text{res}_*$  described in [3, Thm. 6, and following discussion]. The commutativity then follows from [3, Lem. 6]. The formula  $P^{-\gamma} \text{ad}(f) P^\gamma g = P^{-\gamma} [f, P^\gamma g] = P^{-\gamma} f P^\gamma g - P^{-\gamma} P^\gamma g f = P^{-\gamma} f P^\gamma g$  (since  $P^{-\gamma} P^\gamma = 0$ ) implies eq. 5.14. For  $n = 1$  this specializes to

$$\begin{aligned} f_0 \otimes f_1 \mapsto & \tau \sum_{\gamma \in \{\pm\}} (-1)^\gamma P_1^{-\gamma} [P_1^\gamma f_0, f_1] \\ = & \tau(P_1^+([f_0, f_1] - [P_1^+ f_0, f_1]) - P_1^- [P_1^+ f_0, f_1]) \end{aligned}$$

and if  $[f_0, f_1] = 0$  (as would be the case if  $f_0, f_1$  are functions on a variety) this simplifies to eq. 5.15 by using  $P_1^+ + P_1^- = \mathbf{1}$ .  $\square$

Next, we shall relate various  $\phi_{HH}$  for changing cubically decomposed algebras. To clarify the distinction, let us agree to write  $\phi_{HH}^A : HH_n(A) \rightarrow k$  instead of  $\phi_{HH}$  plain.

**Theorem 26** (Local formula). *Suppose  $X/k$  is a reduced finite type scheme of pure dimension  $n$ . Suppose  $\Delta = (\eta_0 > \cdots > \eta_n) \in S_n(X)$  with  $\text{codim}_X \eta_i = i$ . Then there is a canonical finite decomposition*

$$A(\Delta, \mathcal{O}_X) \cong \prod K_j$$

*with each  $K_j$  an  $n$ -local field.*

- (1) *Each  $E_j := \{f \in E_\Delta \mid f K_j \subseteq K_j\}$  with ideals  $J_i^\pm := I_i^\pm \cap E_j$  is a cubically decomposed algebra over  $k$  and for  $f \in HH_n(\mathcal{O}_{\eta_0})$  we have*

$$(5.16) \quad \phi_{HH}^{E_\Delta}(f) = \sum_j \phi_{HH}^{E_j}(f).$$

- (2) *There exists (non-canonically) an isomorphism  $K_j \simeq k'((t_1)) \cdots ((t_n))$  with  $k'/k$  a finite field extension so that*

$$\phi_{HH}^{E_j}(t_1^{c_{0,1}} \cdots t_n^{c_{0,n}} \otimes \cdots \otimes t_1^{c_{n,1}} \cdots t_n^{c_{n,n}}) = (-1)^n \prod_{i=1}^n c_{i,i}$$

*whenever  $\forall i : \sum_{p=0}^n c_{p,i} = 0$  and zero otherwise.*

- (3) *Precomposed with the HKR isomorphism (cf. eq. 5.3), this yields*

$$\Omega_{\mathcal{O}_{\eta_0}/k}^n \longrightarrow HH_n(\mathcal{O}_{\eta_0}) \longrightarrow k$$

$$f_0 df_1 \wedge \cdots \wedge df_n = (-1)^n \det \begin{pmatrix} c_{1,1} & \cdots & c_{n,1} \\ \vdots & \ddots & \vdots \\ c_{1,n} & \cdots & c_{n,n} \end{pmatrix}$$

*for  $f_p = t_1^{c_{p,1}} \cdots t_n^{c_{p,n}}$  ( $0 \leq p \leq n$ ) whenever  $\forall i : \sum_{p=0}^n c_{p,i} = 0$  and zero otherwise.*

*Proof.* Almost all of the first claim follows directly from Prop. 9. **(1)** Observe that the  $E_j$  are unital associative algebras. Define  $J_i^\pm := I_i^\pm \cap E_j$  with  $I_i^\pm$  the ideals of the cubically decomposed algebra structure of  $E_\Delta$ , cf. Prop. 13. It is clear that the  $J_i^\pm$  are two-sided ideals in  $E_j$  and  $J_i^+ + J_i^- = E_\Delta \cap E_j = E_j$ . Since  $J_{tr} = \bigcap_{i=1,\dots,n} \bigcap_{s \in \{\pm\}} J_i^s \subseteq I_{tr}$  we can just use the trace map of  $E_\Delta$ . This proves that  $(E_j, \{J_i^\pm\}, \text{tr}_{I_{tr}})$  is a unital cubically decomposed algebra. In particular, the maps  $\phi_{HH}^{E_j}$  exist. The embedding  $\mathcal{O}_{\eta_0} \hookrightarrow A(\Delta, \mathcal{O}_X) \cong \prod K_j$  is actually diagonal, i.e.  $f \mapsto (f, \dots, f)$ . As a result, the associated multiplication operator in  $E_\Delta$  is diagonal in the  $K_j$ , therefore eq. 5.16 holds. **(2)** For the evaluation of  $\phi_{HH}^{E_j}$  we pick an isomorphism  $K_j \simeq k'((t_1)) \cdots ((t_n))$  (exists by Prop. 9). Henceforth, in order to distinguish clearly between  $t_i$  as a multiplication operator  $x \mapsto t_i \cdot x$ , or as a topological basis element of  $K_j$  as a  $k$ -vector space, we write the latter in bold letters  $\mathbf{t}_i$ . Define idempotents by

$$P_i^+ \mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n} := \delta_{\lambda_i \geq 0} \mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n}$$

and prolong them by continuity to all of  $K_j$  (the  $\mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n}$  only span a vector subspace of countable dimension, but they clearly lie dense). Define  $P_i^- = \mathbf{1} - P_i^+$ . We know that  $\text{im } P_i^+ \subseteq J_i^+$  is a lattice and  $P_i^- (\text{im } P_i^+) = 0$ , so we have a system of good idempotents in the sense of Def. 21. Thus, by Thm. 24 we have  $\phi_{HH}^{E_j}(f_0 \otimes \cdots \otimes f_n) = (-1)^n \text{tr } M$  for the operator  $M$  defined through

$$M := \sum_{\gamma_1 \cdots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \cdots + \gamma_n} P_1^{-\gamma_1} f_1 P_1^{\gamma_1} \cdots P_n^{-\gamma_n} f_n P_n^{\gamma_n} f_0.$$

The remaining computation is essentially the same as in the proof of [3, Thm. 7], so we just sketch the key steps: Letting  $f_k := \mathbf{t}_1^{c_{k,1}} \cdots \mathbf{t}_n^{c_{k,n}}$  for  $c_{k,i} \in \mathbf{Z}$  (and  $0 \leq k \leq n$ ;  $1 \leq i \leq n$ ), one easily computes

$$P_k^- f_k P_k^+ \mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n} = \delta_{0 \leq \lambda_k < -c_{k,k}} \mathbf{t}_1^{\lambda_1 + c_{k,1}} \cdots \mathbf{t}_n^{\lambda_n + c_{k,n}}.$$

This closely mimicks the one-dimensional computation in Lemma 5. With this we can explicitly compute the action of  $M$  on a monomial. We get

$$M \mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n} = \prod_{i=1}^n (\delta_{0 \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{p,i} < -c_{i,i}} - \delta_{-c_{i,i} \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{p,i} < 0})$$

$$\mathbf{t}_1^{\lambda_1 + c_{0,1} + \sum_{p=1}^n c_{p,1}} \cdots \mathbf{t}_n^{\lambda_n + c_{0,n} + \sum_{p=1}^n c_{p,n}}.$$

It is immediately clear that this operator can have a non-zero trace only if  $\forall i : \sum_{p=0}^n c_{p,i} = 0$  holds, because otherwise it is visibly nilpotent and we can invoke axiom **T3** of Tate's trace.

This proves the vanishing part of the claim. Now assume this condition holds and simplify the formula for  $M$  accordingly. A simple eigenvalue count reveals

$$\mathrm{tr} M = \prod_{i=1}^n (-c_{i,i}) = (-1)^n \prod_{i=1}^n c_{i,i}.$$

See the proof of [3, Thm. 7] for the full details. **(3)** For the last claim, plugging in the antisymmetrizer coming from the HKR isomorphism, we get

$$= (-1)^n \sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}(\pi) \prod_{i=1}^n c_{\pi(i),i},$$

which (up to a sign) is exactly the Leibniz formula for the determinant.  $\square$

## 6. A NEW APPROACH

**6.1. Introduction.** We want to change our perspective. Let  $(A, (I_i^\pm), \tau)$  be a cubically decomposed algebra. So far we have always worked in the category of  $A$ -bimodules and considered exact sequences of  $A$ -bimodules like

$$(6.1) \quad 0 \longrightarrow I_n^0 \xrightarrow{\mathrm{diag}} I_n^+ \oplus I_n^- \xrightarrow{\mathrm{diff}} A \longrightarrow 0$$

or their higher-dimensional counterparts as in eq. 5.7. This approach corresponds to viewing Hochschild homology as a functor

$$A\text{-bimodules} \rightarrow k\text{-vector spaces}, \quad M \mapsto H_i(A, M).$$

However, Hochschild homology can also be regarded as a functor

$$\text{associative } k\text{-algebras} \rightarrow k\text{-vector spaces}, \quad A \mapsto HH_i(A).$$

In this section we want to transform the mechanisms of §3, §5 from the former to the latter perspective.

**6.2. Recollections.** We shall need to work with *non-unital* algebras, so let us briefly recall the necessary material (see [34] for details). Hochschild homology was defined and described in §5 for an arbitrary associative algebra  $A$ . We may read  $A$  as a bimodule over itself and if  $A$  is unital we write  $HH_i(A) := H_i(A, A)$ . If  $A$  is not unital, all definitions still make sense and we write  $HH_i^{naiv}(A) := H_i(A, A)$  for these groups, following [20, §1.4.3]. However, this is not a good definition in general, so usually one proceeds differently: There is a unitalization  $A^+$  along with a canonical map  $k \hookrightarrow A^+$  of unital associative algebras, and one defines<sup>6</sup>

$$(6.2) \quad HH_i(A) := \mathrm{coker} \left( HH_i(k) \rightarrow HH_i(A^+) \right),$$

see [20, §1.4] for details; this parallels a similar construction in algebraic  $K$ -theory. If  $A$  happens to be unital, this agrees with the previous definition as in §5, i.e. it agrees with  $HH_i^{naiv}$ . In general, there is only the obvious morphism  $\kappa : HH_i^{naiv}(A) \rightarrow HH_i(A)$  (sending a pure tensor to itself in  $A^+$ ) which need neither be injective nor surjective.

If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is a short exact sequence of  $A$ -bimodules, the sequence  $0 \rightarrow C_\bullet(A, M') \rightarrow C_\bullet(A, M) \rightarrow C_\bullet(A, M'') \rightarrow 0$  is obviously an exact sequence of complexes, so there is a long exact sequence in Hochschild homology

$$(6.3) \quad \cdots \rightarrow H_i(A, M') \rightarrow H_i(A, M) \rightarrow H_i(A, M'') \xrightarrow{\partial} H_{i-1}(A, M') \rightarrow \cdots$$

We denote the connecting homomorphism by  $\partial$ . If  $I$  is a two-sided ideal in  $A$ , this yields the sequence

$$(6.4) \quad \cdots \rightarrow H_i(A, I) \rightarrow H_i(A, A) \xrightarrow{\mu} H_i(A, A/I) \xrightarrow{\partial} H_{i-1}(A, I) \rightarrow H_{i-1}(A, A) \rightarrow \cdots$$

<sup>6</sup>This is not the definition given in our main reference [34]; here  $HH_i(A)$  is the homology of  $\mathcal{K}$ , cf. p. 598, l. 5 in *loc. cit.*, defined in terms of the bar complex. The equivalence of definitions follows from the paragraph before Thm. 3.1 in *loc. cit.*

Moreover, if  $M$  is an  $A/I$ -bimodule, it is also an  $A$ -bimodule via  $A \twoheadrightarrow A/I$ . Then there is an obvious change-of-algebra map  $\nu : C_i(A, M) \rightarrow C_i(A/I, M)$ . Clearly  $A/I$  is an  $A/I$ -bimodule and thus there are canonical maps

$$j : C_i(A, A) \xrightarrow{\mu} C_i(A, A/I) \xrightarrow{\nu} C_i(A/I, A/I),$$

where  $\mu$  is the morphism inducing the respective arrow in eq. 6.4. One also defines the *relative Hochschild homology* complex  $\mathcal{K}_\bullet(A \rightarrow A/I)$ , the precise definition is somewhat involved, see [34, beginning of §3, where instead of  $\mathcal{C}$  one uses the Hochschild version  $\mathcal{K}$ , defined on the same page 598 in line 5]. We write  $HH_i(A \text{ rel } I) := H_i \mathcal{K}_\bullet(A \rightarrow A/I)$  for its homology (*Beware:* The notation  $HH_i(A, I)$  is customary. However, it is easily confused with  $H_i(A, I)$ , which also plays a role here, so we have opted for the present clearer distinction). We may regard  $I$  as an associative algebra itself, but unless  $A = I$  it will not be unital. There is a useful weaker condition than being unital:

**Definition 27.**  $I$  is said to have local units if for every finite set of elements  $a_1, \dots, a_r \in I$  there exists some  $e \in I$  such that  $a_i e = a_i = e a_i$  holds.

**Proposition 28** ([33], [34, Thm. 3.1]). Suppose  $A$  is an associative algebra and  $I$  a two-sided ideal which has local units. Then the canonical morphisms

$$(6.5) \quad HH_i^{\text{naiv}}(I) \xrightarrow{\kappa} HH_i(I) \xrightarrow{\diamond} HH_i(A \text{ rel } I)$$

are both isomorphisms. If  $A$  additionally has local units, there is a quasi-isomorphism

$$(6.6) \quad \mathcal{K}_\bullet(A \rightarrow A/I) \simeq_{\text{qis}} \ker(C_\bullet(A, A) \xrightarrow{j} C_\bullet(A/I, A/I)).$$

It is noteworthy that only the right-most term in eq. 6.5 actually depends on  $A$ .

*Proof.* For the proof, combine [34, Thm. 3.1 and Cor. 4.5] for the first claim: The existence of local units implies  $H$ -unitality. For the second claim,  $A$  is  $H$ -unital, so the bar complex in the definition of  $\mathcal{K}$  in *loc. cit.* p. 598 in line 5 is zero up to quasi-isomorphism. Applying this to the definition of  $\mathcal{K}_\bullet(A \rightarrow A/I)$  in §3 in *loc. cit.* gives the second claim. For an alternative presentation, combine the treatment [20, §1.4.9] with the generality of [20, E.1.4.6]. The  $H$ -unitality of  $A/I$  follows from [34, Cor. 3.4].  $\square$

Basically by construction we get a long exact sequence in homology

$$(6.7) \quad \cdots \rightarrow HH_i(A \text{ rel } I) \rightarrow HH_i(A) \rightarrow HH_i(A/I) \xrightarrow{\delta} HH_{i-1}(A \text{ rel } I) \rightarrow \cdots$$

Although different, it is not unrelated to the sequence in eq. 6.4:

**Lemma 29.** Suppose  $A$  is an associative algebra with local units and  $I$  a two-sided ideal with local units. Then the diagram

$$(6.8) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & H_i(A, I) & \longrightarrow & H_i(A, A) & \longrightarrow & H_i(A, A/I) \xrightarrow{\partial} \cdots \\ & & \downarrow & & \downarrow \kappa & & \downarrow \lambda \\ \cdots & \longrightarrow & HH_i(A \text{ rel } I) & \longrightarrow & HH_i(A) & \longrightarrow & HH_i(A/I) \xrightarrow{\delta} \cdots \end{array}$$

is commutative.

*Proof.* Trivial if  $A$  is unital. In general: We construct this on the level of complexes  $C_\bullet(-, -)$ . The middle downward arrow maps pure tensors to themselves,  $A \rightarrow A^+$  in  $HH_i(A^+)$  and then to the cokernel as given by eq. 6.2. Similarly, the right-hand side downward arrow is induced by

$$a_0 \otimes a_1 \otimes \cdots \otimes a_i \mapsto a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_i},$$

where  $a_0 \in A/I$ ,  $a_1, \dots, a_n \in A$  and  $\tau : A \twoheadrightarrow A/I$  is the quotient map, again sent to  $(A/I)^+$  and then to the respective cokernel. For the left-hand side we can wlog. use the presentation on the right-hand side of eq. 6.6 for  $HH_i(A \text{ rel } I)$ . The downward arrow is then given by the analogous formula, but  $a_0 \in I$  and so  $\overline{a_0} = 0$  in  $A/I$ , so that it is clear that the image lies in the kernel of  $j : C_i(A, A) \rightarrow C_i(A/I, A/I)$ .  $\square$

### 6.3. The construction.

**Definition 30.** We say that an  $n$ -fold cubically decomposed algebra  $(A, (I_i^\pm), \tau)$  has local units on all levels if  $A$  and any intersection  $I_{i_1}^{s_{i_1}} \cap \dots \cap I_{i_r}^{s_{i_r}}$  made from the ideals  $\{I_1^\pm, \dots, I_n^\pm\}$  has local units.

Let  $(A^n, (I_i^\pm), \tau)$  be such an  $n$ -fold cubically decomposed algebra over  $k$ . Define

$$(6.9) \quad A^{n-1} := I_n^0 \quad J_i^\pm := I_i^\pm \cap A^{n-1} \quad (\text{for } i = 0, \dots, n-1).$$

Then  $(A^{n-1}, (J_i^\pm), \tau)$  is an  $(n-1)$ -fold cubically decomposed algebra over  $k$ . The truth of Definition 30 clearly persists: If the former has local units on all levels, so has the latter. Evaluating eq. 6.9 inductively, we find  $A^s = (I_{s+1}^0 \cap \dots \cap I_n^0) \cap A$ . Define

$$(6.10) \quad \Lambda : A^n \longrightarrow A^n / A^{n-1}, \quad x \longmapsto x^+$$

where  $x = x^+ + x^-$  is any decomposition with  $x^\pm \in I_n^\pm$  (always exists and gives well-defined map). This map does *not* equal the natural quotient map! The definition of  $\Lambda$  might seem a little artificial, but it arises entirely naturally, see Remark 5.

*Example 7.* If we consider the Laurent polynomial ring  $k[t, t^{-1}] = \bigoplus_{i \in \mathbb{Z}} k \cdot t^i$  and take for  $A$  the infinite matrix algebra acting on it, cf. Example 6, the multiplication operator  $t^i : \mathbf{t}^\lambda \mapsto \mathbf{t}^{\lambda+i}$  satisfies  $\Lambda t^i \cdot \mathbf{t}^\lambda = \delta_{\lambda+i \geq 0} \mathbf{t}^{\lambda+i}$ . This suggests to view  $\Lambda t^i$  as a kind of Toeplitz operator.

Using the relative Hochschild homology sequence, eq. 6.7, coming from the exact sequence of associative algebras

$$(6.11) \quad 0 \longrightarrow A^{n-1} \longrightarrow A^n \xrightarrow{\text{quot}} A^n / A^{n-1} \longrightarrow 0,$$

the connecting homomorphism induces a map  $\delta$  and we employ it to define a map

$$(6.12) \quad d : HH_{i+1}(A^n) \xrightarrow{\Lambda} HH_{i+1}(A^n / A^{n-1}) \xrightarrow{\delta} HH_i(A^{n-1}).$$

We can repeat this construction and obtain a morphism:

**Definition 31.** Suppose  $(A, (I_i^\pm), \tau)$  is an  $n$ -fold cubically decomposed algebra over  $k$  which has local units on all levels. Then there is a canonical map

$$\phi_C : HH_n(A) \longrightarrow HH_0(I_{tr}) \longrightarrow k, \quad \alpha \mapsto \tau \underbrace{d \circ \dots \circ d}_n \alpha.$$

Analogously, for cyclic homology  $\phi_C : HC_n(A) \rightarrow k$  (see lemma below why we call this  $\phi_C$  as well).

**Lemma 32.** The map  $\phi_C$  factors over  $HH_n(A) \xrightarrow{I} HC_n(A) \longrightarrow k$ .

*Proof.* Let  $d'$  be the analogue of the map in eq. 6.12 with cyclic homology. Both  $\Lambda$  and the connecting map are compatible with  $I$  so that

$$\begin{array}{ccc} HH_n(A) & \xrightarrow{d \circ \dots \circ d} & HH_0(I_{tr}) \\ \downarrow I & & \downarrow I \\ HC_n(A) & \xrightarrow{d' \circ \dots \circ d'} & HC_0(I_{tr}) \end{array}$$

commutes, but the right-hand side downward arrow is an isomorphism, giving the claim.  $\square$

We would like to stress that apart from our set of axioms, almost nothing goes into the construction of this map, it is a very simple definition. Let me reiterate that  $\Lambda$  is not the quotient map, otherwise  $d$  would clearly just be the zero map.

We are now ready for the main comparison theorem:

**Theorem 33.** *Suppose  $(A, (I_i^\pm), \tau)$  is a unital  $n$ -fold cubically decomposed algebra over  $k$  which has local units on all levels. Then  $\phi_C : HH_n(A) \rightarrow k$  agrees up to sign with  $\phi_{HH}$ , namely*

$$\phi_C = (-1)^{\frac{n(n-1)}{2}} \phi_{HH}.$$

*Proof.* (1) We proceed by induction. Firstly, we construct a commutative diagram and a map  $\Psi$ :

$$(6.13) \quad \begin{array}{ccccc} H_s(A, A^s) & \longleftarrow & H_s(A^s, A^s) & \xrightarrow{\kappa} & HH_s(A^s) \\ \downarrow \Lambda & & \downarrow \Lambda & & \downarrow \Lambda \\ H_s(A, \frac{A^s}{A^{s-1}}) & \longleftarrow & H_s(A^s, \frac{A^s}{A^{s-1}}) & \xrightarrow{\lambda} & HH_s(\frac{A^s}{A^{s-1}}) \\ \downarrow \partial & & \downarrow \partial & & \downarrow \delta \\ H_{s-1}(A, A^{s-1}) & \longleftarrow & H_{s-1}(A^s, A^{s-1}) & \longrightarrow & HH_{s-1}(A^{s-1}) \end{array}$$

$\Psi$    $d$

The leftward arrows are the change-of-algebra maps along  $A^s \hookrightarrow A$ . The commutativity of the upper left square is immediate, the one on the right agrees with the rightmost square in Lemma 29. The downward arrows in the middle row come from the connecting homomorphism in the long exact sequences (as in eq. 6.3 and eq. 6.7, combined with Wodzicki excision) arising from eq. 6.11. The commutativity of the lower squares then follows from Lemma 29 (all involved algebras have local units by Def. 30). (2) Next, we patch the outer columns of the diagram as in eq. 6.13 for  $s = n, n-1, \dots, 1$  under each other, giving

$$\begin{array}{ccc} H_n(A, A^n) & \xleftarrow{\cong} & H_n(A^n, A^n) \xrightarrow{\cong} HH_n(A^n) \\ \downarrow \Psi & & \downarrow d \\ H_{n-1}(A, A^{n-1}) & & HH_{n-1}(A^{n-1}) \\ \vdots & & \vdots \\ H_0(A, A^0) & \longleftarrow & H_0(A^1, A^0) \longrightarrow HH_0(A^0) \end{array}$$

The middle column of the previous diagram does not fit to be glued into this pattern, so we omit it, except for the top and bottom row. The morphisms in the top row are isomorphisms since  $A$  (unlike the  $A^s$  for  $s < n$ ) is unital. We evaluate the terms in the lowest row and compose with the trace  $\tau$ , giving the diagram

$$\begin{array}{ccc} \frac{A^0}{[A, A^0]} & \longleftarrow & \frac{A^0}{[A^1, A^0]} \longrightarrow HH_0(A^0) \\ \downarrow & & \downarrow \\ k & \xleftarrow{\cong} & k \xrightarrow{\cong} k. \end{array}$$

Since the trace  $\tau$  factors through  $[A, A^0]$  (note that  $A^0 = I_{tr}$ ), it is clear that the arrows in the bottom row must be isomorphisms. Thus,  $\phi_C = \tau d^{cn} = \tau \Psi^{cn}$ . Note that this comparison only

works because in the top and bottom row all terms are isomorphic, whereas on the intermediate rows it is not clear whether there should exist arrows from the left to the right column (or reversely). It remains to compute  $\tau\Psi^{on}$ :

(3) Consider the diagram with exact rows

$$(6.14) \quad \begin{array}{ccccc} I_s^0 & \xrightarrow{\text{diag}} & I_s^+ \oplus I_s^- & \longrightarrow & A^s \\ \downarrow = & & \downarrow pr_{I_s^+} & & \downarrow (1) \\ A^{s-1} & \longrightarrow & I_s^+ & \longrightarrow & A^s/I_s^- \\ \downarrow = & & \downarrow \text{incl} & & \downarrow (2) \\ A^{s-1} & \longrightarrow & A^s & \longrightarrow & A^s/A^{s-1} \end{array} \quad \Lambda$$

(here for readability we have omitted intersecting all the ideals with  $A^s$ ; everything is understood to be subobjects of  $A^s$ ). The map  $pr_{I_s^+}$  is the projection  $(x^+, x^-) \mapsto x^+$ . Pick the arrows (1) and (2) so that the diagram becomes commutative. We find both are given by  $x \mapsto x^+$  where  $x = x^+ + x^-$  with  $x^\pm \in I_s^\pm$  is any decomposition of  $x$ . Moreover, the composition on the right is indeed  $\Lambda$ . Taking the long exact sequences in Hochschild homology of the top and bottom row yields

$$\begin{array}{ccccc} H_s(A, I_s^+ \oplus I_s^-) & \longrightarrow & H_s(A, A^s) & \xrightarrow{\partial} & H_{s-1}(A, I_s^0) \\ \text{incl} \downarrow pr_{I_s^+} & & \downarrow \Lambda & & \downarrow \cong \\ H_s(A, A^s) & \longrightarrow & H_s(A, A^s/A^{s-1}) & \xrightarrow{\partial} & H_{s-1}(A, A^{s-1}) \end{array}$$

Now by the commutativity of the above diagram  $\Psi = \partial \circ \Lambda : H_s(A, A^s) \rightarrow H_{s-1}(A, A^{s-1})$  (as on the left in eq. 6.13) can be computed just by unwinding the connecting map in the top row. It stems from the bimodule exact sequence in the top row of eq. 6.14: Evaluating this is an easy chase of the snake map, compare with the proof of Lemma 5: Pick some system of good idempotents. We need to pick a lift of  $a_0 \otimes a_1 \otimes \cdots \otimes a_s \in C_s(A, A^s)$  to  $C_s(A, I_s^+ \oplus I_s^-)$ . We may take  $f_\gamma := (-1)^\gamma P_s^\gamma a_0 \otimes a_1 \otimes \cdots \otimes a_s$  for  $\gamma \in \{\pm\}$  respectively. We need to apply the differential  $b$ , resulting in

$$\begin{aligned} bf_\gamma &= (-1)^\gamma (P_s^\gamma a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_s + \sum_{j=1}^{s-1} (-1)^j P_s^\gamma a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_s \\ &\quad + (-1)^s a_s P_s^\gamma a_0 \otimes a_1 \otimes \cdots \otimes a_{s-1}) \in C_{s-1}(A, I_1^\gamma) \end{aligned}$$

Next, we need to determine the preimage in  $C_{s-1}(A, I_s^0) = C_{s-1}(A, A^{s-1})$ , which is

$$\begin{aligned} \Psi(a_0 \otimes \cdots \otimes a_s) &= \sum_{\gamma \in \{\pm\}} (-1)^\gamma P_s^{-\gamma} (bf_\gamma) \\ &= (-1)^s \left( \sum_{\gamma \in \{\pm\}} (-1)^\gamma P_s^{-\gamma} a_s P_s^\gamma \right) a_0 \otimes a_1 \otimes \cdots \otimes a_{s-1}. \end{aligned}$$

Hence, by applying this formula inductively, we get

$$\tau\Psi^{on}(a_0 \otimes \cdots \otimes a_n) = (-1)^{1+2+\cdots+n} \tau \prod_{s=1 \dots n} \left( \sum_{\gamma \in \{\pm\}} (-1)^\gamma P_s^{-\gamma} a_s P_s^\gamma \right) a_0.$$

This expression clearly coincides (up to sign) with the one of Theorem 24 so that the previously proven identity  $\phi_C = \tau d^{on} = \tau\Psi^{on}$  implies the claim.  $\square$

*Remark 5.* The key point of the diagram in eq. 6.14 is the transition from sequences of bimodules to ring extensions. Note that  $I_s^0$  is not an ideal in  $I_s^+ \oplus I_s^-$ , so we cannot apply Lemma 29 to



the top row just like that. Having the diagram in eq. 6.14 induced from the left and middle column, the map  $\Lambda$  arises naturally on the right to make it commute.

*Remark 6.* (Alternative approach) Instead of the comparison of  $H_s(A, A^s)$  and  $HH_s(A^s)$  in eq. 6.13 we could have tried to work with the groups  $HH_s(-)$  only. However, then the long exact sequence in homology reads

$$\cdots \longrightarrow HH_s(A^s/A^{s-1}) \xrightarrow{\delta} HH_{s-1}(A^s \text{ rel } A^{s-1}) \longrightarrow HH_{s-1}(A^s) \longrightarrow \cdots$$

with the relative Hochschild homology group as described in eq. 6.6 and in order to get an explicit formula for the map, we would need to make the inverse  $\diamond^{-1}$  of Wodzicki's excision isomorphism explicit, cf. Prop. 28. Although clearly being the more direct approach, this seems to lead to a far lengthier computation.

**Corollary 34.** *Under the assumptions of the theorem and  $\mathfrak{g} := A_{Lie}$  the diagram*

$$\begin{array}{ccccc} H_n(\mathfrak{g}, \mathfrak{g}) & \xrightarrow{\varepsilon} & HH_n(A) & \xrightarrow{\phi_{HH}} & k \\ I' \downarrow & & I \downarrow & & \downarrow = \\ H_{n+1}(\mathfrak{g}, k) & \xrightarrow{(-1)^n \varepsilon} & HC_n(A) & \xrightarrow{\phi_C} & k \end{array}$$

commutes, where in the lower row

$$\varepsilon(f_0 \wedge \cdots \wedge f_n) := \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) f_0 \otimes f_{\pi^{-1}(1)} \otimes \cdots \otimes f_{\pi^{-1}(n)}$$

for  $f_0, \dots, f_n \in \mathfrak{g}$ . The composed map  $H_n(\mathfrak{g}, \mathfrak{g}) \rightarrow k$  agrees with  $H_n(\mathfrak{g}, \mathfrak{g}) \xrightarrow{I'} H_n(\mathfrak{g}, k) \xrightarrow{\phi_{Beil}} k$ .

*Proof.* The left-hand side square commutes by direct inspection. Then combine Cor. 25 and Cor. 32.  $\square$

**Acknowledgement.** *The original ideas belong to John Tate and Alexander Beilinson. This paper is just a lengthy interpretation and elaboration of their beautiful works [1], [32], possibly adding mistakes, but perhaps a way of expressing appreciation for their inventiveness.*

*I thank Ivan Fesenko for explaining the problem to me; Michael Groechenig and Jesse Wolfson for many interesting discussions on the ind-pro approach; Amnon Yekutieli for his many insightful remarks, suggestions and careful reading of an earlier version. This has been most helpful.*

*Moreover, I thank the Essen Seminar for Algebraic Geometry and Arithmetic for the excellent working conditions and friendly atmosphere.*

## REFERENCES

- [1] A. Beilinson. Residues and adèles. *Funktsional. Anal. i Prilozhen.*, 14(1):44–45, 1980.
- [2] A. Beilinson, B. Feigin, and B. Mazur. Notes on conformal field theory. unpublished, available on <http://www.math.sunysb.edu/~kirillov/manuscripts.html>, 1991.
- [3] O. Braunling. Adele residue symbol and Tate's central extension for multiloop Lie algebras. to appear in *Algebra and Number Theory* 2014, 2012.
- [4] O. Braunling, M. Groechenig, and J. Wolfson. Tate Objects in Exact Categories. arXiv:1402.4969 [math.KT], 2014.
- [5] A. Cámara. Functional analysis on two-dimensional local fields. *Kodai Math. J.*, 36(3):536–578, 2013.
- [6] B. Conrad. *Grothendieck duality and base change*, volume 1750 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.
- [7] I. Fesenko and M. Kurihara, editors. *Invitation to higher local fields*, volume 3 of *Geometry & Topology Monographs*. Geometry & Topology Publications, Coventry, 2000. Papers from the conference held in Münster, August 29–September 5, 1999.

- [8] I. Fesenko and S. Vostokov. *Local fields and their extensions*, volume 121 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, second edition, 2002. With a foreword by I. R. Shafarevich.
- [9] M. Gerstenhaber and S. D. Schack. A Hodge-type decomposition for commutative algebra cohomology. *J. Pure Appl. Algebra*, 48(3):229–247, 1987.
- [10] S. Gorchinskiy and A. Rosly. A polar complex for locally free sheaves. *International Mathematics Research Notices*, 2014.
- [11] R. Hartshorne. *Residues and duality*. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. *Lecture Notes in Mathematics*, No. 20. Springer-Verlag, Berlin, 1966.
- [12] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [13] A. Huber. Adèle für Schemata und Zariski-Kohomologie. In *Schriftenreihe des Mathematischen Instituts der Universität Münster*, 3. Serie, Heft 3, volume 3 of *Schriftenreihe Math. Inst. Univ. Münster 3. Ser.*, page 86. Univ. Münster, Münster, 1991.
- [14] A. Huber. On the Parshin-Beilinson adèles for schemes. *Abh. Math. Sem. Univ. Hamburg*, 61:249–273, 1991.
- [15] R. Hübl. Residues of regular and meromorphic differential forms. *Math. Ann.*, 300(4):605–628, 1994.
- [16] R. Hübl and A. Yekutieli. Adèles and differential forms. *J. Reine Angew. Math.*, 471:1–22, 1996.
- [17] J. Lipman. *Residues and traces of differential forms via Hochschild homology*, volume 61 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1987.
- [18] J. Lipman. Notes on derived functors and Grothendieck duality. In *Foundations of Grothendieck duality for diagrams of schemes*, volume 1960 of *Lecture Notes in Math.*, pages 1–259. Springer, Berlin, 2009.
- [19] J. Lipman and M. Hashimoto. *Foundations of Grothendieck duality for diagrams of schemes*, volume 1960 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.
- [20] J.-L. Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco.
- [21] A. Neeman. The Grothendieck duality theorem via Bousfield’s techniques and Brown representability. *J. Amer. Math. Soc.*, 9(1):205–236, 1996.
- [22] D. Osipov. Adeles on  $n$ -dimensional schemes and categories  $C_n$ . *Internat. J. Math.*, 18(3):269–279, 2007.
- [23] D. Osipov.  $n$ -dimensional local fields and adeles on  $n$ -dimensional schemes. In *Surveys in contemporary mathematics*, volume 347 of *London Math. Soc. Lecture Note Ser.*, pages 131–164. Cambridge Univ. Press, Cambridge, 2008.
- [24] F. Pablos Romo. On the linearity property of Tate’s trace. *Linear Multilinear Algebra*, 55(4):323–326, 2007.
- [25] A. Parshin. On the arithmetic of two-dimensional schemes. I. Distributions and residues. *Izv. Akad. Nauk SSSR Ser. Mat.*, 40(4):736–773, 949, 1976.
- [26] A. Parshin. Chern classes, adèles and  $L$ -functions. *J. Reine Angew. Math.*, 341:174–192, 1983.
- [27] A. Parshin and T. Fimmel. Introduction to higher adelic theory (draft). unpublished, 1999.
- [28] L. Previdi. Locally compact objects in exact categories. *Internat. J. Math.*, 22(12):1787–1821, 2011.
- [29] P. Sastry and A. Yekutieli. On residue complexes, dualizing sheaves and local cohomology modules. *Israel J. Math.*, 90(1-3):325–348, 1995.
- [30] J.-P. Serre. Faisceaux algébriques cohérents. *Ann. of Math. (2)*, 61:197–278, 1955.
- [31] J.-P. Serre. *Algebraic groups and class fields*. Graduate Texts in Mathematics, Springer Verlag New York, 1997.
- [32] J. Tate. Residues of differentials on curves. *Ann. Sci. École Norm. Sup. (4)*, 1:149–159, 1968.
- [33] M. Wodzicki. The long exact sequence in cyclic homology associated with an extension of algebras. *C. R. Acad. Sci. Paris Sér. I Math.*, 306(9):399–403, 1988.
- [34] M. Wodzicki. Excision in cyclic homology and in rational algebraic  $K$ -theory. *Ann. of Math. (2)*, 129(3):591–639, 1989.
- [35] A. Yekutieli. An explicit construction of the Grothendieck residue complex. *Astérisque*, (208):127, 1992. With an appendix by Pramathanath Sastry.
- [36] A. Yekutieli. Residues and differential operators on schemes. *Duke Math. J.*, 95(2):305–341, 1998.
- [37] A. Yekutieli. The action of adeles on the residue complex. *Comm. Algebra*, 31(8):4131–4151, 2003. Special issue in honor of Steven L. Kleiman.

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, THEA-LEYMANN-STRASSE 9, 45127 ESSEN, GERMANY

*E-mail address:* oliver.braunling@uni-due.de